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Topological pressure of proper map

Dongkui Ma*, Nuanni Fan

School of Mathematics, South China University of Technology, Guangzhou, 510641, PR China

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ABSTRACT

Based on the Carathéodory–Pesin structure theory [13], we introduce three notions of topological pressure of a proper map and provide some properties of these notions. For the proper map of a locally compact separable metric space, we prove some variational principles and give some applications on the multifractal analysis of local entropies. These are the extensions of results of Pesin, Takens and Verbiski, etc.

1. Introduction

Let f be a continuous map acting on a compact metric space X and φ a continuous function on X. The notion of topological pressure of φ was brought to the theory of dynamical systems by Ruelle [14] and Walters [17], and it was further developed by Pesin and Pitskel [13]. The topological pressure is a key notion in dynamical systems and dimension theory. In [12], Pesin used the dimension approach to the notion of topological pressure, which was based on the Carathéodory structure [6](we call it the Carathéodory–Pesin structure, or briefly, C–P structure). It is a very powerful tool to study dimension theory and dynamical systems. For a proper map, Patrão [11], Ma and Cai [10] introduced some notions of topological entropy. Moreover, Li and Zhang [8,9] introduced nonadditive and almost additive topological pressures.

In this paper, by using the C–P structure, we introduce three notions of topological pressure for a proper map of a metric space. They are extensions of the classical topological pressures introduced by Walters [17], Pesin and Pitskel [13] respectively. Some properties of these notions are provided. For the proper map of a locally compact separable metric space, we prove some variational principles and give some applications on the multifractal analysis of local entropies. These extend some results of Pesin [12], Takens and Verbitski [15], etc.

This paper is organized as follows. In Section 2, we introduce the notions of the topological pressure, the lower and upper capacity topological pressure and give some basic properties of them. In Section 3, we give some further properties. In Section 4, we give some variational principles. In Section 5, we give some applications.

2. Topological pressure, lower and upper capacity topological pressures and their basic properties

In this section, by using the C–P structure [12], the topological pressure and lower and upper capacity topological pressure are introduced for the proper map of a metric space.

Let X be a topological space and f: $X \to X$ the proper maps, i.e., f is a continuous map such that the pre-image by f of any compact set is compact. An open set is called an admissible open set if the closure or the complement of it is compact. An admissible cover of X is an open and finite cover $\mathcal{U}$ of X such that, for each $U \in \mathcal{U}$, U is an admissible open set.

Let (X, d) be a metric space and denote $B(x, \delta)$ the open ball centered at x with radius $\delta > 0$. The metric d is called admissible [11] if the following conditions are satisfied:

1. If $\mathcal{U}_\delta = \{B(x_1, \delta), \ldots, B(x_k, \delta)\}$ is a cover of X, for every $\delta \in (a, b)$, where $0 < a < b$, then there exists $\delta_\varepsilon \in (a, b)$ such that $\mathcal{U}_{\delta_\varepsilon}$ is admissible.
2. Every admissible cover of X has a Lebesgue number.

* Corresponding author.
E-mail addresses: dkma@scut.edu.cn (D. Ma), fannuanni@126.com (N. Fan).

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In [10], the authors proved that for a metric space (X, d), every admissible cover of X has a Lebesgue number, so the condition (2) in the definition of admissible metric can be deleted. From [11], we see that if d is an admissible metric, then for any $\varepsilon > 0$ there exists an admissible cover such that the diameter of this cover is less than ε. It is easy to see that, if (X, d) is compact, then d is automatically admissible.

Let (X, d) be a metric space and f: $X \to X$ a proper map. Given an admissible cover $\mathcal{U}$ of X, denote by $\mathcal{S}_m(\mathcal{U})$ the set of all strings $\mathbf{U} = (U_{i_0}, U_{i_1}, \dots, U_{i_{m-1}})$ of length $m = m(\mathbf{U})$, where $U_{i_j} \in \mathcal{U}, j = 0, 1, \dots, m-1$. We put $\mathcal{S} = \mathcal{S}(\mathcal{U}) = \bigcup_{m\ge 0} \mathcal{S}_m(\mathcal{U})$.

To a given string $\mathbf{U} = (U_{i_0}, U_{i_1}, \dots, U_{i_{m-1}}) \in \mathcal{S}(\mathcal{U})$ we associate the set

$$X(\mathbf{U}) = \{x \in X : f^j(x) \in U_{i_j}, j = 0, 1, \dots, m(U) - 1\}.$$

It is easy to see that $X(\mathbf{U}) = \bigcap_{j=0}^{m(U)-1} f^{-j}U_{i_j}$, and $X(\mathbf{U})$ is an admissible open set. Let $\varphi \in C(X, \mathbb{R})$ be bounded, where $C(X, \mathbb{R})$ denotes the space of real-valued continuous functions of X. Denote $(S_n\varphi)(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$. Define the collection of subsets

$$\mathcal{F} = \mathcal{F}(\mathcal{U}) = \{X(\mathbf{U}) : \mathbf{U} \in \mathcal{S}(\mathcal{U})\}$$

and three functions $\xi, \eta, \psi : \mathcal{S}(\mathcal{U}) \to \mathbb{R}^+$ as follows

$$\xi(\mathbf{U}) = \exp\left(\sup_{x\in X(\mathbf{U})} (S_{m(\mathbf{U})}\varphi)(x)\right),$$
$$\eta(\mathbf{U}) = \exp(-m(\mathbf{U})),$$
$$\psi(\mathbf{U}) = m(\mathbf{U})^{-1}.$$

It is easy to verify that the sets $\mathcal{S}, \mathcal{F}$ and the functions ξ, η, and ψ determine a C–P structure $\tau = \tau(\mathcal{U}) = (S, \mathcal{F}, \xi, \eta, \psi)$ on X (see [12]). We say that a collection of strings $\mathcal{G}$ covers a set $Z \subset X$ if $\bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) \supset Z$. For any set $Z \subset X$ and $\alpha \in \mathbb{R}$, define

$$\begin{aligned} &M(Z, \alpha, \varphi, \mathcal{U}, N) \\ &:= \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \xi(\mathbf{U})\eta(\mathbf{U})^\alpha \right\} \\ &= \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(-\alpha m(\mathbf{U}) + \sup_{x\in X(\mathbf{U})} (S_{m(\mathbf{U})}\varphi)(x)\right)\right\}, \end{aligned}$$

and the infimum is taken over all finite or countable collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) \ge N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. Let

$$m(Z, \alpha, \varphi, \mathcal{U}) = \lim_{N\to+\infty} M(Z, \alpha, \mathcal{U}, N).$$

For every real numbers α introduce

$$\underline{r}(Z, \alpha, \varphi, \mathcal{U}) = \underline{\lim}_{N\to\infty} R(Z, \alpha, \varphi, \mathcal{U}, N),$$

$$\overline{r}(Z, \alpha, \varphi, \mathcal{U}) = \overline{\lim}_{N\to\infty} R(Z, \alpha, \varphi, \mathcal{U}, N),$$

where

$$R(Z, \alpha, \varphi, \mathcal{U}, N) = \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(-\alpha N + \sup_{x\in X(\mathbf{U})} (S_N\varphi)(x)\right)\right\},$$

and the infimum is taken over all collections of strings $\mathcal{G} \subset S(\mathcal{U})$ such that $m(\mathbf{U}) = N$ for all $\mathbf{U} \in \mathcal{G}$ and $\mathcal{G}$ covers Z. By the definition of C–P structure, define

$$\begin{aligned} P_Z(\varphi, \mathcal{U}) &:= \inf\{\alpha : m(Z, \alpha, \varphi, \mathcal{U}) = 0\} \\ &= \sup\{\alpha : m(Z, \alpha, \varphi, \mathcal{U}) = \infty\}, \\ \underline{CP}_Z(\varphi, \mathcal{U}) &:= \inf\{\alpha : \underline{r}(Z, \alpha, \varphi, \mathcal{U}) = 0\} \\ &= \sup\{\alpha : \underline{r}(Z, \alpha, \varphi, \mathcal{U}) = \infty\}, \\ \overline{CP}_Z(\varphi, \mathcal{U}) &:= \inf\{\alpha : \overline{r}(Z, \alpha, \varphi, \mathcal{U}) = 0\} \\ &= \sup\{\alpha : \overline{r}(Z, \alpha, \varphi, \mathcal{U}) = \infty\}. \end{aligned}$$

Lemma 2.1. *[([11])] Let (X, d) be a metric space, then every admissible cover of X has a Lebesgue number.*

Theorem 2.2. *Let (X, d) be a metric space and d an admissible metric, f: $X \to X$ a proper map, $\varphi \in C(X, \mathbb{R})$ bounded. Then for any $Z \subset X$, the following limits exist:*

$$P_Z(\varphi) := \lim_{|\mathcal{U}|\to 0} P_Z(\varphi, \mathcal{U}),$$

$$\underline{CP}_Z(\varphi) := \lim_{|\mathcal{U}|\to 0} \underline{CP}_Z(\varphi, \mathcal{U}),$$

$$\overline{CP}_Z(\varphi) := \lim_{|\mathcal{U}|\to 0} \overline{CP}_Z(\varphi, \mathcal{U}),$$

where $\mathcal{U}$ is admissible cover and $|\mathcal{U}|$ denotes the diameter of $\mathcal{U}$, i.e., $|\mathcal{U}| = \max\{diam(\mathbf{U}) : \mathbf{U} \in \mathcal{U}\}$.

Proof. We use the similar method as that of [12]. By Lemma 2.1, we let $\mathcal{V}$ be an admissible cover of X with diameter smaller than the Lebesgue number of admissible cover $\mathcal{U}$. One can see that each element $V \in \mathcal{V}$ is contained in some element $U(V) \in \mathcal{U}$. To any string $\mathbf{V} = (V_{i_0}, \dots, V_{i_m}) \in \mathcal{S}(\mathcal{V})$ we associate the string $\mathbf{U}(\mathbf{V}) = (U(V_{i_0}), \dots, U(V_{i_m})) \in \mathcal{S}(\mathcal{U})$. If $\mathcal{G} \subset \mathcal{S}(\mathcal{V})$ covers a set $Z \subset X$ then $\mathbf{U}(\mathcal{G}) = \{\mathbf{U}(\mathbf{V}) : \mathbf{V} \in \mathcal{G}\} \subset \mathcal{S}(\mathcal{U})$ also covers Z. Let $\gamma = \gamma(\mathcal{U}) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in U, U \in \mathcal{U}\}$. Then for every $\alpha \in \mathbb{R}$ and $N > 0$

$$M(Z, \alpha, \varphi, \mathcal{U}, N) \le M(Z, \alpha - \gamma, \varphi, \mathcal{V}, N).$$

We deduce that

$$P_Z(\varphi, \mathcal{U}) - \gamma \le P_Z(\varphi, \mathcal{V}).$$

Since X has admissible cover of arbitrarily small diameter, then

$$P_Z(\varphi, \mathcal{U}) - \gamma \le \underline{\lim}_{|\mathcal{V}|\to 0} P_Z(\varphi, \mathcal{V}).$$

If $|\mathcal{U}| \to 0$, then $\gamma(\mathcal{U}) \to 0$ and hence

$$\overline{\lim}_{|\mathcal{U}|\to 0} P_Z(\varphi, \mathcal{U}) \le \underline{\lim}_{|\mathcal{V}|\to 0} P_Z(\varphi, \mathcal{V}).$$

This implies the existence of the first limit. The others can be proved in a similar fashion. □

We call the quantities $P_Z(\varphi)$, $\overline{CP}_Z(\varphi, f)$, $\underline{CP}_Z(\varphi)$ respectively, the topological pressure and lower and upper capacity topological pressure of the function φ on the set Z(with respect to f). We write $P_{Z,f}(\varphi)$, $\overline{CP}_{Z,f}(\varphi, f)$, $\underline{CP}_{Z,f}(\varphi)$ respectively to emphasize f if we need to.

By the basic properties of the C–P structure [12], we get the following basic properties. From Theorem 2.3 to Theorem 2.7, we always assume that (X, d) is a metric space with d being an admissible metric, f: $X \to X$ is a proper map and $\varphi, \psi \in C(X, \mathbb{R})$ is bounded.

Theorem 2.3.

1. $P_\emptyset(\varphi) \le 0$.
2. $P_{Z_1}(\varphi) \le P_{Z_2}(\varphi)$ *if* $Z_1 \subset Z_2 \subset X$.
3. $P_Z(\varphi) = \sup_{i\ge 1} P_{Z_i}(\varphi)$, *where* $Z = \bigcup_{i\ge 1} Z_i$, $Z_i \subset X, i = 1, 2, \dots$.
4. *If f is a homeomorphism then* $P_Z(\varphi) = P_{f(Z)}(\varphi)$, *where Z is any subset of X.*

Theorem 2.4.

1. $\underline{CP}_\emptyset(\varphi) \le 0$, $\overline{CP}_\emptyset(\varphi) \le 0$;
2. $\underline{CP}_{Z_1}(\varphi) \le \underline{CP}_{Z_2}(\varphi)$, $\overline{CP}_{Z_1}(\varphi) \le \overline{CP}_{Z_2}(\varphi)$ *if* $Z_1 \subset Z_2 \subset X$.
3. $\underline{CP}_Z(\varphi) \ge \sup_{i\ge 1} \underline{CP}_{Z_i}(\varphi)$ *and* $\overline{CP}_Z(\varphi) \ge \sup_{i\ge 1} \overline{CP}_{Z_i}(\varphi)$, *where* $Z = \bigcup_{i\ge 1} Z_i$, $Z_i \subset X, i = 1, 2, \dots$.