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Bi-center conditions and local bifurcation of critical periods in a switching Z_2 equivariant cubic system

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ABSTRACT

In this study, we consider bi-centers and local bifurcation of critical periods for a switching Z_2 equivariant cubic system. We give the necessary and sufficient conditions for the system to have bi-centers at the symmetrical singular points $(\pm 1, 0)$. We develop a method for computing the period constants near the center of switching systems and use this method to study bifurcation of critical periods for a switching system. We further find the existence of 10 local critical periods bifurcating from these bi-centers.

1. Introduction

The dynamical systems whose right-hand sides are discontinuous or non-differentiable often encountered in different fields of mechanics, engineering and automatic control, see [1–3]. These systems are called discontinuous or non-smooth systems. After the work of Filippov [4], in the context of planar Filippov systems, there has been attracted increasing interest in the qualitative analysis of discontinuous systems. One class of discontinuous systems is the so-called switching system, which has different definitions of the continuous vector fields in two different regions divided by at least one switching line or curve. We consider a discontinuous differential system of linear center-focus mode switching line on the x-axis as follows

$$(\dot{x}, \dot{y}) = \begin{cases} \left(\delta x - y + \sum_{k=2}^{n} X_k^{+}(x, y),\ x + \delta y + \sum_{k=2}^{n} Y_k^{+}(x, y)\right), & \text{for } \ y > 0, \\ \left(\delta x - y + \sum_{k=2}^{n} X_k^{-}(x, y),\ x + \delta y + \sum_{k=2}^{n} Y_k^{-}(x, y)\right), & \text{for } \ y < 0, \end{cases} \tag{1}$$

where $\delta \in \mathbf{R}$, $X_k^{\pm}(x, y)$ and $Y_k^{\pm}(x, y)$ are homogeneous polynomials with respect to x and y, $k = 2, 3, \ldots, n$. Our attention in system (1) has two systems. The first equation is called the upper system which is defined for $y > 0$, and the second equation is called the lower system which is defined for $y < 0$.

The studies on center-focus problem for switching systems were started in [5,6]. Recently, many results have been obtained on the Hopf bifurcation in switching systems, see [7–12]. Unlike analytic systems, the switching systems of same degree may produce more limit cycles. Coll, Gasull and Prohens [7] introduced three different types of singular points, called focus-focus (FF), parabolic-focus (PF) and parabolic-parabolic (PP) for switching systems. They proved that a switching quadratic system has at least 4 limit cycles in the FF-type case. For the switching linear systems, Han and Zhang [9] showed that 2 limit cycles can be produced from a focus of either FF, FP or PP type. Chen et al. [10] constructed a class of switching quadratic Bautin systems with 8 limit cycles bifurcating from the center. In [11], Tian and Yu gave a new method with an efficient algorithm for computing the Lyapunov constants of planar switching systems. And they gave an example of switching Bautin systems showing the existence of 10 limit cycles bifurcating from the origin. Li et al. [12] proved that a switching cubic system can have 15 limit cycles bifurcating from three equilibria. Using Mel-

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nikov's method, the author of [13] investigated limit cycle bifurcations for a class of perturbed planar switching systems with 4 switching lines.

The conditions for the equilibrium to be a center and an isochronicity center in switching systems are quite complex. As indicated in [11,12,14], the origin of system (1) can be a center even if it is not a center of either the upper system or the lower one. On the other hand, the inverse proposition is uncertain. Center conditions were obtained for some switching Kukles systems [8], switching Liénard systems [15] and a switching Bautin system [16]. Li and Liu [17] investigated the center conditions and limit cycles in a class of switching systems with a degenerate singular point. Chen and Zhang [14] discussed the problem of center and isochrony center in a particular Bautin switching system that the singular is a center of both the upper system and the lower one. Some researches of isochronicity center in switching systems were studied in [12,18]. The local critical period of planar differential systems is a problem connected tightly with the center and the limit cycle. In 1989, Chicone and Jacobs introduced the notion of bifurcation of local critical periods by analogy with the method of Bautin [19]. More information about critical periods can be found in [20–23]. Yu et al. [22] showed the existence of 7 local critical periods for third-order planar Hamiltonian systems, which is the maximum number of local critical periods that a cubic system can have. As far as we know, there are few papers concerning the problem of local critical periods in switching system (1).

The problem of determining the number of limit cycles and bifurcation of critical periods are determined by orders of the Lyapunov constants and period constants respectively. It is difficult to get the desired result by using single focus point for a high-degree polynomial system. Yu and Han [24,25] proved that some smooth Z_2 equivariant cubic systems having 2 elementary foci can have 12 small-amplitude limit cycles with symmetry. The authors of [26,27] presented some results about the limit cycle bifurcations in Z_n equivariant systems. In [28], Li and Liu investigated the following Z_2 equivariant cubic system

$$\begin{aligned}\dot{x} &= -(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3,\\ \dot{y} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3,\end{aligned} \tag{2}$$

with parameter $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbf{R}^6$. It has two finite elementary foci which is arranged at $(1, 0)$ and $(-1, 0)$. They showed that system (2) can have 13 limit cycles and eleven conditions of $(1, 0)$ or $(-1, 0)$ to be a center. It is the best result about the number of limit cycles for smooth cubic systems. For other lower bounds of the number of limit cycles for smooth differential systems, see [29,30]. If a center condition holds, it is called $(1, 0)$ or $(-1, 0)$ a bi-center. Romanovski et al. [31] investigated the simultaneous existence of bi-centers and their isochronicity for two families of Z_2 equivariant cubic systems. The authors of [23] obtained the maximum number of local critical periods bifurcating from bi-centers of system (2) is 6. In this paper, we study bi-center conditions and local bifurcation of critical periods for a switching Z_2 equivariant cubic system. Without loss of generality, the switching Z_2 equivariant cubic system can be written as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases}\begin{pmatrix}-(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3\\ -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3\end{pmatrix}, & \text{for } y > 0,\\ \begin{pmatrix}-(b_1+1)y + b_1x^2y + b_2xy^2 + b_3y^3\\ -\frac{1}{2}x - b_4y + \frac{1}{2}x^3 + b_4x^2y + b_5xy^2 + b_6y^3\end{pmatrix}, & \text{for } y < 0.\end{cases} \tag{3}$$

The paper is organized as follows. In the next section, we introduce the procedure to compute the Lyapunov constants at the singular point and develop a procedure to compute period constants near center of the general system (1), which only involves algebraic computations. In Section 3, we study the bi-center conditions in a switching Z_2 equivariant cubic system, obtained by taking $a_4 = b_4 = 0$ in (3)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{cases}\begin{pmatrix}-(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3\\ -\frac{1}{2}x + \frac{1}{2}x^3 + a_5xy^2 + a_6y^3\end{pmatrix}, & \text{for } y > 0,\\ \begin{pmatrix}-(b_1+1)y + b_1x^2y + b_2xy^2 + b_3y^3\\ -\frac{1}{2}x + \frac{1}{2}x^3 + b_5xy^2 + b_6y^3\end{pmatrix}, & \text{for } y < 0.\end{cases} \tag{4}$$

We only consider those cases that the singular points $(1, 0)$ and $(-1, 0)$ are bi-centers of the upper system in (4), i.e., system (4) with $\lambda = (a_1, a_2, a_3, a_5, a_6, b_1, b_2, b_3, b_5, b_6) \in \mathbf{R}^{10}$ in the union of the sets

$$D_i = \{(\lambda_1, \cdots, \lambda_5, \mu_1, \cdots, \mu_5) \in \mathbf{R}^{10} \,|\, (\lambda_1, \cdots, \lambda_5) \in B_i\},\quad 1 \le i \le 4, \tag{5}$$

where

$$\begin{aligned}B_1 &= \left\{(\lambda_1, \cdots, \lambda_5) \in \mathbf{R}^5 \,\middle|\, \lambda_1 = -\lambda_5, \lambda_2 = -3\lambda_6\right\};\\ B_2 &= \left\{(\lambda_1, \cdots, \lambda_5) \in \mathbf{R}^5 \,\middle|\, \lambda_1 + \lambda_5 \neq 0, \lambda_2 = \lambda_6 = 0\right\};\\ B_3 &= \Big\{(\lambda_1, \cdots, \lambda_5) \in \mathbf{R}^5 \,\Big|\, \lambda_3 = 2(1+\lambda_1)(1+\lambda_5),\\ &\qquad \lambda_6 = \frac{1}{3}(-\lambda_2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_5), (2+\lambda_1+\lambda_5)\lambda_2 = 0\Big\};\\ B_4 &= \Big\{(\lambda_1, \cdots, \lambda_5) \in \mathbf{R}^5 \,\Big|\, \lambda_5 \neq 1, \lambda_6 = \frac{1}{3}(\lambda_2 - 2\lambda_2\lambda_5),\\ &\qquad \lambda_1 = -1, \lambda_3 = 0\}.\end{aligned} \tag{6}$$

From [28, Theorem 2.5], we obtain that the singular points $(1, 0)$ and $(-1, 0)$ of the upper system of system (4) are bi-centers if and only if $(a_1, a_2, a_3, a_5, a_6)$ lies in $B_1 \cup B_2 \cup B_3 \cup B_4$. We present the necessary and sufficient conditions for the existence of bi-centers for system (4). All possible first integrals are given. In Section 4, we prove that the maximum number of local critical period bifurcations at bi-center $(1, 0)$ (or $(-1, 0)$) of system (4) is 5, with a concrete numerical example to illustrate the existence of 5 critical period. Hence, system (4) has exact 10 local critical periods bifurcating from bi-centers $(1, 0)$ and $(-1, 0)$. This is a new result for such cubic systems concerning the number of local critical periods.

2. Lyapunov constants and period constants

For analytic system, the Lyapunov constants are important to solve center problems. First, we present some basic formulae of computing the Lyapunov constants at the singular point of the switching system (1). Under the polar coordinates transformation, $x = r\cos\theta$ and $y = r\sin\theta$, system (1) can be transformed as

$$\dot{r} = \begin{cases}\delta r + \sum_{k=2}^{n} \Upsilon_k^+(\theta) r^k, & \text{for } \theta \in (0, \pi),\\ \delta r + \sum_{k=2}^{n} \Upsilon_k^-(\theta) r^k, & \text{for } \theta \in (\pi, 2\pi),\end{cases} \tag{7}$$

$$\dot{\theta} = \begin{cases}1 + \sum_{k=2}^{n} \Theta_k^+(\theta) r^{k-1}, & \text{for } \theta \in (0, \pi),\\ 1 + \sum_{k=2}^{n} \Theta_k^-(\theta) r^{k-1}, & \text{for } \theta \in (\pi, 2\pi),\end{cases} \tag{8}$$

where

$$\begin{aligned}\Upsilon_k^{\pm}(\theta) &= \cos\theta X_k^{\pm}(\cos\theta, \sin\theta) + \sin\theta Y_k^{\pm}(\cos\theta, \sin\theta),\\ \Theta_k^{\pm}(\theta) &= \cos\theta Y_k^{\pm}(\cos\theta, \sin\theta) - \sin\theta X_k^{\pm}(\cos\theta, \sin\theta).\end{aligned} \tag{9}$$