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Anomalous diffusion resulted from fractional damping

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ABSTRACT

The dynamics of a Brownian particle diffusing in the pure fractional damping environment is studied in the field of a metastable potential. Several anomalous behaviors are revealed such as that a type of reverse diffusion is encountered. And the diffusion dynamics is closely related to the fractional exponent α. Particles are found to move reversely in the opposite direction of diffusion when α is relatively large despite of the zero-approximating effective friction of the system

1. Introduction

The phenomenon of anomalous diffusion is ubiquitous in various classical and quantum dissipative systems [1–3]. Previous studies have found that it may primarily be resulted from the nontrivial type of coupling between the system and its reservoir [4,5] or some kind of non-Markovian effect of the friction. Therefore, the characterization of the friction kernel is always a crucial point to understand anomalous diffusion by using of the generalized Langevin equation.

In an early study of R. Muralidhar and his collaborators, a type of power-law friction kernel has been used to approximately describe the anomalous diffusion on two-dimensional critical percolation clusters [6], which gives us some insight on how to characterize the friction kernel as an effective constitutive property of the fractal medium. Recently, we renamed this kind of friction as fractional damping and successfully elucidated the quantum thermodynamic properties of the system under its affection [7]. This encouraged us to concern more about the anomalous dynamics resulted from fractional damping.

Actually, fractional damping has been observed in a multitude of systems such as amorphous semiconductors [8], subsurface tracer dispersion [9,10] and financial dynamics [11]. The subdiffusive behaviors relate to it is also quite abundant in small systems. Typical instances include the motion of small probe beads in actin networks [12], the propagation of virus shells in cells [13] and the motion of telomeres in mammalian cells [14], etc.

However, little has been known about the dynamical details of a Brownian particle diffusing in the pure fractional damping environment. This may be caused by the difficult fractional calculus immersed in the process of theoretical analysis. Although it has past more than 320 years since the first proposition of fractional calculus in 1695 by L. Hospital, the mathematical theory of it remains imperfect. In particular, the development of fractional derivative makes slow progress, resulting great difficulties in the solving of fractional differential equations.

Therefore in this paper, we report one of our recent study on this point. In Section 2, kinematic relations of the diffusion particles are obtained by Laplacian solving the fractional generalized Langevin equation. The probability of successfully escaping from the potential well is computed in Section 3 for a characteristic visualization. Section 4 serves as a short summary of present results where some discussions are also made for a further consideration.

2. Fractional Langevin equation and its solution

Mathematically, the motion of a Brownian particle with mass m diffusing in the pure fractional damping environment can be described by the following fractional generalized Langevin equation (GLE) [15–17]

$$m\ddot{x} + \int_0^t \eta(t-t')\dot{x}(t')dt' + \partial_x U(x) = \xi(t), \tag{1}$$

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where $\eta(t) = \eta_\alpha t^{-\alpha}/\Gamma(1-\alpha)$ is the fractional friction kernel identified by a fractional exponent α ranging from 0 to 1. η_α is a constant introduced to indicate the friction strength resulted from the viscosity and $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function. $\xi(t)$ is the fractional Gaussian noise with zero mean and its second momentum satisfying the fluctuation-dissipation theorem $\langle \xi(t)\xi(t')\rangle = k_BT\eta_\alpha|t-t'|^{-\alpha}$ [18].

The solution of GLE, namely also the equation of motion for the diffusing particle, can be easily obtained by a series of Laplacian transformation. For an example, in the particular case of an inverse harmonic potential $U(x) = -\frac{1}{2}m\omega^2x^2$, we find after some algebra

$$x(t) = \langle x(t)\rangle + \int_0^t H(t-\tau)\xi(\tau)d\tau, \quad (2)$$

in which the mean position of the particle along the transport direction is given by

$$\langle x(t)\rangle = \left[1+\omega^2\int_0^t H(\tau)d\tau\right]x_0 + H(t)v_0, \quad (3)$$

and the variance of $x(t)$ reads

$$\sigma_x^2(t) = \int_0^t dt_1 H(t-t_1)\int_0^{t_1} dt_2\langle\xi(t_1)\xi(t_2)\rangle H(t-t_2), \quad (4)$$

where $H(t) = \mathcal{L}^{-1}[(s^2+s\eta(s)-\omega^2)^{-1}]$ is namely the response function and $\eta(s) = \eta_\alpha s^{\alpha-1}$ is the Laplacian transformation of the friction kernel $\eta(t)$.

3. Diffusion dynamics of fractional damping system

For the study of diffusion dynamics, one of the basic task is to compute the successful rate of the particle escaping from a metastable potential well. Mathematically, it always emerges to be a complementary error function due to the Gaussian property of noise $\xi(t)$ and linearity of the GLE. i.e.

$$P(t) = \int_0^\infty W(x,t)dx = \frac{1}{2}\mathrm{erfc}\left(-\frac{\langle x(t)\rangle}{\sqrt{2A_{11}(t)}}\right), \quad (5)$$

yielding from a short integration on the reduced distribution function $W(x, t)$ where

$$W(x,t) = \frac{1}{\sqrt{2\pi A_{11}(t)}}\exp\left[-\frac{(x-\langle x(t)\rangle)^2}{2A_{11}(t)}\right], \quad (6)$$

is resulted from the joint probability density function (PDF) of the system [19]

$$W(x,v,t) = \frac{1}{2\pi|\mathbf{A}(t)|^{1/2}}e^{-\frac{1}{2}[y^\dagger(t)\mathbf{A}^{-1}(t)y(t)]}, \quad (7)$$

with $y(t)$ the vector $[x-\langle x(t)\rangle, v-\langle v(t)\rangle]$ and $\mathbf{A}(t)$ the matrix of second moments

$$A_{11}(t) = \sigma_x^2(t) = \langle[x-\langle x(t)\rangle]^2\rangle, \quad (8a)$$

$$A_{12}(t) = A_{21}(t) = \langle[x-\langle x(t)\rangle][v-\langle v(t)\rangle]\rangle, \quad (8b)$$

$$A_{22}(t) = \sigma_v^2(t) = \langle[v-\langle v(t)\rangle]^2\rangle. \quad (8c)$$

In the calculations here and following, we rescale all the quantities so that dimensionless unit such as $k_BT = 1.0$ is used. The metastable potential $U(x)$ is approximated to be an inverse harmonic potential in the neighbourhood of its saddle point. Firstly in Figs. 1 and 2, we plot the time dependent varying of $\langle x(t)\rangle$ and $A_{11}(t)$ (namely also $\sigma_x^2(t)$) at various α. From which we can see that $\langle x(t)\rangle$ increases quickly when α is relatively small. But as the increasing of α, $\langle x(t)\rangle$ decays suddenly into a negative diverging

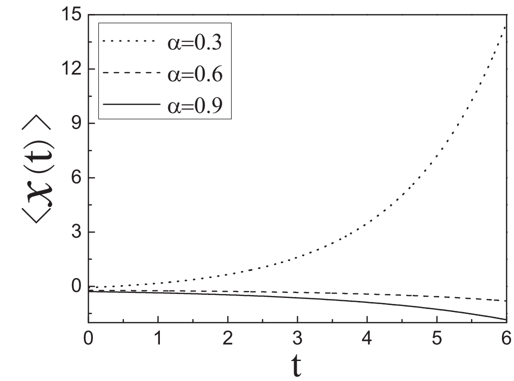

Fig. 1. Time dependent varying of $\langle x(t)\rangle$ at various α. Parameters in use are $\eta_\alpha = 2.0$, $\omega = k_BT = 1.0$, $x_0 = -1.0$ and $v_0 = 2.0$.

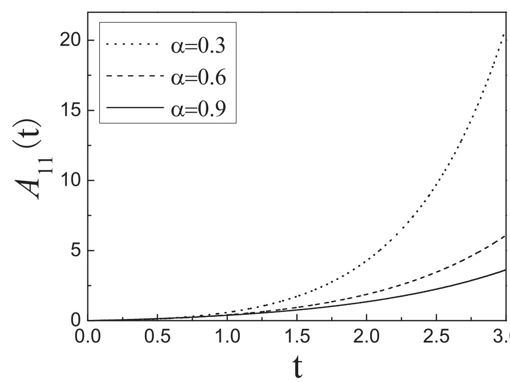

Fig. 2. Time dependent varying of $A_{11}(t)$ at various α.

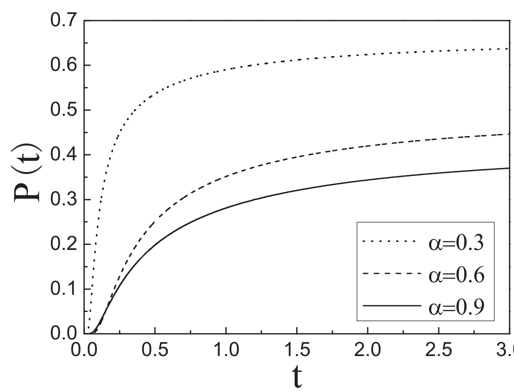

Fig. 3. Time dependent varying of $P(t)$ at various α.

function of α. This is a very nontrivial result, because from the view point of distribution evolution, $\langle x(t)\rangle$ indicates the center of the PDF wave packets and $A_{11}(t)$ the width. Therefore it reveals that the center of the PDF may move in the opposite direction of diffusion.

However, the value of $A_{11}(t)$ is always positively increasing. This means, as the forwarding of the center, the width of the PDF wave packet is continuously expanding. Therefore, despite of the unusual movement of the center one may always expect a steady barrier escaping probability. As is shown in Fig. 3, the probability for a particle to pass over the saddle point tends to be a steady one in the long time limit no matter what is the value of α. The occurrence of such a nontrivial result may probably be caused by the property of long-range correlations immersed in the fractional damping environment. And this reveals from another point of view that the fractional damping environment is in analogy with the