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Fractional analysis of co-existence of some types of chaos synchronization

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ABSTRACT

Chaotic dynamics and synchronization of fractional-order systems have attracted much attention recently. Based on stability theory of fractional-order systems and stability theory of integer-order systems, this paper deals with the problem of coexistence of various types of synchronization between different dimensional fractional-order chaotic systems. To illustrate the capabilities of the novel schemes proposed herein, numerical and simulation results are given.

1. Introduction

Synchronization, one of the most universal collective rhythms, has attracted a great deal of interest among researchers from various fields due to its potential applications in physics and engineering since the pioneering work by Pecora and Carroll [1]. The research efforts have been devoted to chaos control and chaos synchronization problems in nonlinear science because of its extensive applications [2–6]. In most real systems, the synchronization is carried out even though the oscillators have different orders. One example is the synchronization that occurs between heart and lung [7], where one can observe that both circulatory and respiratory systems behave in synchronous way, but their models are essentially different and they have different order. So, the study of synchronization for strictly different dynamical systems and different order dynamical systems is both very important from the perspective of control theory and very necessary from the perspective of practical application.

Similar to a nonlinear differential system, nonlinear fractional differential system may also have complex dynamics such as chaos and bifurcation. Recent studies have shown efforts to explore synchronization behavior of coupled fractional-order systems [8–11]. However, it is difficult to explicitly construct Laypunov function for fractional differential systems. Therefore, in many literatures, synchronization among fractional-order systems is only investigated through numerical simulations that are based on the stability criteria of linear fractional-order systems, such as the work presented in [12–18], or based on Laplace transform theory, such as the work presented in [19–21]. Fractional chaos synchronization has great potential applications in secure communication and cryptography [22,23]. Furthermore, some hardware implementations of fractional-order systems have been proposed in the literature [24–26].

Recently, an interesting type of synchronization between chaotic and hyperchaotic systems has been introduced [27], in which each master system state synchronizes with a linear combination of slave system states. The proposed scheme is called inverse full state hybrid projective synchronization (IFSHPS). Also, a new type of generalized synchronization to synchronize nonidentical chaotic systems with different dimensions has been proposed in [28]. This type of synchronization was called inverse generalized synchronization (IGS). IGS is characterized by the existence of a functional relationship φ between the state Y of the slave system and the state X of the master system, so that $X = \varphi(Y)$ after

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a transient time. Not long ago, a new approach to synchronize different dimensional chaotic system by using two scaling matrices was introduced. The method was called $\Theta - \Phi$ synchronization [29,30]. In [31], $\Theta - \Phi$ synchronization was used to synchronize different dimensional fractional-order chaotic systems by using scaling constant matrix and scaling function matrix.

When studying the synchronization of chaotic systems, an interesting phenomenon that may occur is the co-existence of several synchronization types. Recently, the problem of coexistence of different types of synchronization has been introduced in many studies [32–38]. The co-existence of synchronization types is very useful in secure communication and chaotic encryption schemes. Based on these considerations, this paper presents new approaches to rigorously study the co-existence of inverse full state hybrid projective synchronization (IFSHPS), $\Theta - \Phi$ synchronization, inverse generalized synchronization (IGS) and Q-S synchronization between chaotic and hyperchaotic fractional-order systems. By using fractional-order Lyapunov stability theorem and Lyapunov stability theory of integer-order systems, the paper, first, analyzes the proposed synchronization scheme when the master system is an incommensurate fractional system and the slave system is a 4-D commensurate fractional system. Successively, stability theory of fractional-order systems and stability theory of linear integer-order 3-D systems are used to prove the mentioned synchronization scheme between 3-D incommensurate fractional master system and 4-D incommensurate fractional slave system. Numerical examples of co-existence of synchronization types are illustrated, with the aim to show the effectiveness of the novel approaches developed herein.

2. Preliminaries

To discuss fractional differential systems, some definitions and properties of Caputo fractional differential operator and some results on the stability of fractional differential systems from the literature which are relevant to our work are introduced.

Definition 1. The Riemann–Liouville fractional integral operator of order $p > 0$ of the function $f(t)$ is defined as [39],

$$J^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau, \quad t > 0. \tag{1}$$

where Γ denotes Gamma function.

Definition 2. The Caputo fractional derivative of $f(t)$ is defined as [40,41],

$$D_t^p f(t) = J^{m-p}\left(\frac{d^m}{dt^m} f(t)\right) = \frac{1}{\Gamma(m-p)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{p-m+1}} d\tau, \tag{2}$$

for $m-1 < p \leq m$, $m \in \mathbf{N}$, $t > 0$.

Lemma 1. *The Laplace transform of the Caputo fractional derivative rule reads [42]*

$$\mathbf{L}\left(D_t^p f(t)\right) = s^p \mathbf{F}(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0),$$
$$(p > 0, \quad n-1 < p \leq n), \tag{3}$$

where $\mathbf{L}(f(t)) = \mathbf{F}(s) = \int_0^\infty e^{-st} f(t) dt$. *Particularly, when* $0 < p \leq 1$, *we have*

$$\mathbf{L}\left(D_t^p f(t)\right) = s^p \mathbf{F}(s) - s^{p-1} f(0). \tag{4}$$

Lemma 2. *The Laplace transform of the Riemann–Liouville fractional integral rule satisfies [43]*

$$\mathbf{L}(J^q f(t)) = s^{-q} \mathbf{F}(s), \quad (q > 0). \tag{5}$$

Lemma 3. *The fractional-order linear system [44]*

$$D_t^p X(t) = AX(t), \tag{6}$$

where $D_t^p = \left[D_t^{p_1}, D_t^{p_2}, \ldots, D_t^{p_n}\right], 0 < p_i \leq 1, \quad (i = 1, 2, \ldots, n)$, $X(t) = (x_i(t))_{1 \leq i \leq n}$ *and* $A \in \mathbf{R}^{n \times n}$, *is asymptotically stable if all roots* λ *of the characteristic equation*

$$\det\left(\mathrm{diag}\left(\lambda^{Mp_1}, \lambda^{Mp_2}, \ldots, \lambda^{Mp_n}\right) - A\right) = 0, \tag{7}$$

satisfy $|\arg(\lambda)| > \frac{\pi}{2M}$, *where M is the least common multiple of the denominators of* p_i*'s.*

Lemma 4. *The trivial solution of the following fractional-order system [45]*

$$D_t^p X(t) = F(X(t)), \tag{8}$$

where $D_t^p = \left[D_t^p, D_t^p, \ldots, D_t^p\right]$, $0 < p \leq 1$, *and F:* $\mathbf{R}^n \to \mathbf{R}^n$, *is asymptotically stable if there exists a positive definite Lyapunov function V such that* $D_t^p V(X(t)) < 0$, *for all* $t > 0$.

Lemma 5. *Let* $X(t) = (x_i(t))_{1 \leq i \leq n} \in \mathbf{R}^n$, *where* $x_i(t)$ *be continuous and derivable function for* $i = 1, 2, \cdots, n$. *Then* $\forall p \in]0, 1]$ *and* $\forall t > 0$ *[46]*

$$\frac{1}{2} D_t^p \left(X^T(t) X(t)\right) \leq X^T(t) D_t^p (X(t)). \tag{9}$$

3. Definitions of IFSHPS, $\Theta - \Phi$ synchronization, IGS and Q-S synchronization

Consider the following master and slave systems

$$D_t^p X(t) = F(X(t)), \tag{10}$$

$$D_t^q Y(t) = G(Y(t)) + U, \tag{11}$$

where $X(t) = (x_i(t))_{1 \leq i \leq n}$ and $Y(t) = (y_i(t))_{1 \leq i \leq m}$ are the states vector of the master system (10) and the slave system (11), respectively, $D_t^p = \left[D_t^{p_1}, D_t^{p_2}, \ldots, D_t^{p_n}\right]$ and $D_t^q = \left[D_t^{q_1}, D_t^{q_2}, \ldots, D_t^{q_m}\right]$ are Caputo fractional derivative operators, where p_i $(i = 1, 2, \ldots, n)$ and q_i $(i = 1, 2, \ldots, m)$ are rational numbers between 0 and 1, F: $\mathbf{R}^n \to \mathbf{R}^n$, G: $\mathbf{R}^m \to \mathbf{R}^m$ and $U = (u_i)_{1 \leq i \leq m}$ is a control law.

Definition 3. The master system (10) and the slave system (11) are said to be inverse full state hybrid projective synchronized (IFSHPS) if there exists controllers u_i, $i = 1, 2, \ldots, m$, and given real numbers α_j, $j = 1, 2, \ldots, m$, such that the synchronization errors

$$e_i(t) = \sum_{j=1}^{m} \alpha_j y_j(t) - x_i(t), \quad i = 1, 2, \ldots, n, \tag{12}$$

satisfy that $\lim_{t \to \infty} e_i(t) = 0$.

Definition 4. The master system (10) and the slave system (11) are said to be $\Theta - \Phi$ synchronized in the dimension d if there exists a controller $U = (u_i)_{1 \leq i \leq m}$, a constant matrix $\Theta = \left(\Theta_{ij}\right)_{d \times m}$ and a function matrix $(\Phi_{ij}(t))_{d \times n}$ such that the synchronization error

$$e(t) = \Theta Y(t) - \Phi(t) X(t), \tag{13}$$

satisfies that $\lim_{t \to \infty} e(t) = 0$.

Definition 5. The master system (10) and the slave system (11) are said to be inverse generalized synchronized if there exists a controller $U = (u_i)_{1 \leq i \leq m}$ and given a differentiable function φ: $\mathbf{R}^m \to \mathbf{R}^n$ such that the synchronization error

$$e(t) = \varphi(Y(t)) - X(t), \tag{14}$$

satisfies that $\lim_{t \to \infty} e(t) = 0$.

Definition 6. The master system (10) and the slave system (11) are said to be Q-S synchronized in the dimension d if there exists a