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p-adic dynamical systems of (2,2)-rational functions with unique fixed point

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ABSTRACT

We consider a family of (2, 2)-rational functions given on the set of complex p-adic field $\mathbb{C}_p$. Each such function has a unique fixed point. We study p-adic dynamical systems generated by the (2, 2)-rational functions. We show that the fixed point is indifferent and therefore the convergence of the trajectories is not the typical case for the dynamical systems. Siegel disks of these dynamical systems are found. We obtain an upper bound for the set of limit points of each trajectory, i.e., we determine a sufficiently small set containing the set of limit points. For each (2, 2)-rational function on $\mathbb{C}_p$ there are two points $\hat{x}_1 = \hat{x}_1(f),\ \hat{x}_2 = \hat{x}_2(f) \in \mathbb{C}_p$ which are zeros of its denominator. We give explicit formulas of radiuses of spheres (with the center at the fixed point) containing some points such that the trajectories (under actions of f) of the points after a finite step come to $\hat{x}_1$ or $\hat{x}_2$. Moreover for a class of (2, 2)-rational functions we study ergodicity properties of the dynamical systems on the set of p-adic numbers Q_p. For each such function we describe all possible invariant spheres. We show that if $p \geq 3$ then the p-adic dynamical system reduced on each invariant sphere is not ergodic with respect to Haar measure. In case $p = 2$ under some conditions we prove non ergodicity and we show that there exists a sphere on which our dynamical system is ergodic.

1. Introduction

We study dynamical systems generated by a rational function. A function is called a (n, m)-rational function if and only if it can be written in the form $f(x) = \frac{P_n(x)}{Q_m(x)}$, where $P_n(x)$ and $Q_m(x)$ are polynomial functions with degree n and m respectively, $Q_m(x)$ is not the zero polynomial.

It is known that analytic functions play a fundamental role in complex analysis and rational functions play an analogous role in p-adic analysis [12,27]. It is therefore natural to study dynamics generated by rational functions in p-adic analysis. In this paper we consider (2, 2)-rational functions on the field of complex p-adic numbers and study behavior of trajectories of the dynamical systems generated by such functions.

The p-adic dynamical systems arise in Diophantine geometry in the constructions of canonical heights, used for counting rational points on algebraic varieties over a number field, as in [10]. Moreover p-adic dynamical systems are effective in computer science (straight line programs), in numerical analysis and in simulations (pseudorandom numbers), uniform distribution of sequences, cryptography (stream ciphers, T-functions), combinatorics (Latin squares), automata theory and formal languages, genetics. The monograph [5] contains the corresponding survey (see also [1,6–30] for the theory and applications of p-adic dynamical systems).

Let us briefly mention papers which are devoted to dynamical systems of (n, m)-rational functions (this is not a complete review of p-adic dynamical systems of rational functions). A polynomial function can be considered as a $(n, 0)$-rational function (see for example, [13]). Therefore, we start from review of such functions. The most studied discrete p-adic dynamical systems (iterations of maps) are the so-called monomial systems.

In [3,16] the behavior of a p-adic dynamical system $f(x) = x^n$ in the fields of p-adic numbers Q_p and $\mathbb{C}_p$ were studied.

In [2] the properties of the nonlinear p-adic dynamic system $f(x) = x^2 + c$ with a single parameter c (i.e., a (2, 0)-rational function) on the integer p-adic numbers $\mathbb{Z}_p$ are investigated. This dynamic system illustrates possible brain behaviors during remembering.

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In [4] (see also [5, Section 4.7]) a general criterion of ergodicity for $\mathcal{A}$-function on spheres is given, which is an important class of p-adic locally analytic functions. The class $\mathcal{A}$ contains a wide subclass of rational functions. This ergodicity criterion then was extended in [8,9] for the class of all non-expansive transformations of the space of 2-adic integers.

In recently published papers [11,19] a criteria of ergodicity of general rational dynamical systems in Q_p are established.

In [15], dynamical systems defined by the functions $f_q(x) = x^n + q(x)$, where the perturbation $q(x)$ is a polynomial whose coefficients have small p-adic absolute value, was studied.

In [22,25] the dynamical systems associated with the function $f(x) = x^3 + ax^2$ on the set of p-adic numbers is studied. More general form of this function, i.e., $f(x) = x^{2n+1} + ax^{n+1}$, is considered in [24].

Papers [14,23] (see also references therein) are devoted to (1, 1)-rational p-adic dynamical systems.

In [1] and [17] the trajectories of an arbitrary (2, 1)-rational p-adic dynamical systems in a complex p-adic field $\mathbb{C}_p$ are studied.

The paper [29] is devoted to a (3, 2)-rational p-adic dynamical system in $\mathbb{C}_p$, when there exists a unique fixed point.

In [30] we continued investigation of the (3, 2)-rational p-adic dynamical systems in $\mathbb{C}_p$, when there are two fixed points.

In this paper we investigate behavior of trajectory of a (2, 2)-rational p-adic dynamical system in $\mathbb{C}_p$.

The paper is organized as follows: in Section 2 we give some preliminaries. Section 3 contains the definition of the (2, 2)-rational function and main results about behavior of trajectories of the p-adic dynamical system. Siegel disks of these dynamical systems are studied. We obtain an upper bound for the set of limit points of each trajectory. We give explicit formulas of radiuses of spheres, with the center at the fixed point, containing some points such that the trajectories of the points after a finite step come to zeros of the denominator of the rational function. In Section 4 for a class of (2, 2)-rational functions we study ergodicity properties of the dynamical systems on the set of p-adic numbers Q_p. For each such function we describe all possible invariant spheres. We study ergodicity of each p-adic dynamical system with respect to Haar measure reduced on each invariant sphere. For $p \geq 3$ it is proved that the dynamical systems are not ergodic. But for $p = 2$ under some conditions the dynamical system may be ergodic.

2. Preliminaries

2.1. p-adic numbers

Let Q be the field of rational numbers. The greatest common divisor of the positive integers n and m is denotes by (n, m). Every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number.

The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

It has the following properties:

1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,
2) $|xy|_p = |x|_p |y|_p$,
3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

3.1) if $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$,
3.2) if $|x|_p = |y|_p$ then $|x + y|_p \leq |x|_p$,

this is a non-Archimedean one.

The completion of Q with respect to p-adic norm defines the p-adic field which is denoted by Q_p (see [18]).

The algebraic completion of Q_p is denoted by $\mathbb{C}_p$ and it is called *complex p-adic numbers*. For any $a \in \mathbb{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathbb{C}_p : |x - a|_p < r\}, \quad V_r(a) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathbb{C}_p : |x - a|_p = r\}.$$

A function $f : U_r(a) \to \mathbb{C}_p$ is said to be *analytic* if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} f_n (x - a)^n, \quad f_n \in \mathbb{C}_p,$$

which converges uniformly on the ball $U_r(a)$.

2.2. Dynamical systems in $\mathbb{C}_p$

Recall some known facts concerning dynamical systems (f, U) in $\mathbb{C}_p$, where f: $x \in U \to f(x) \in U$ is an analytic function and $U = U_r(a)$ or $\mathbb{C}_p$ (see for example [26]).

Now let f: $U \to U$ be an analytic function. Denote $f^n(x) = \underbrace{f \circ \dots \circ f}_{n}(x)$.

If $f(x_0) = x_0$ then x_0 is called a *fixed point*. The set of all fixed points of f is denoted by Fix(f). A fixed point x_0 is called an *attractor* if there exists a neighborhood $U(x_0)$ of x_0 such that for all points $x \in U(x_0)$ it holds $\lim_{n\to\infty} f^n(x) = x_0$. If x_0 is an attractor then its *basin of attraction* is

$$A(x_0) = \{x \in \mathbb{C}_p : \ f^n(x) \to x_0, \ n \to \infty\}.$$

A fixed point x_0 is called *repeller* if there exists a neighborhood $U(x_0)$ of x_0 such that $|f(x) - x_0|_p > |x - x_0|_p$ for $x \in U(x_0)$, $x \neq x_0$.

Let x_0 be a fixed point of a function $f(x)$. Put $\lambda = f'(x_0)$. The point x_0 is attractive if $0 < |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and repelling if $|\lambda|_p > 1$.

The ball $U_r(x_0)$ (contained in V) is said to be a *Siegel disk* if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $f^n(x) \in S_\rho(x_0)$ for all $n = 1, 2 \dots$. The union of all Siegel desks with the center at x_0 is said to *a maximum Siegel disk* and is denoted by $SI(x_0)$.

3. (2, 2)-Rational *p*-adic dynamical systems

In this paper we consider the dynamical system associated with the (2, 2)-rational function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by

$$f(x) = \frac{ax^2 + bx + c}{x^2 + dx + e}, \quad a \neq 0, \quad |b - ad|_p + |c - ae|_p \neq 0, \quad a, b, c, d, e \in \mathbb{C}_p. \tag{3.1}$$

where $x \neq x_{1,2} = \frac{-d \pm \sqrt{d^2 - 4e}}{2}$.

Remark 1. We note that if $b = ad$ and $c = ae$ then from (3.1) we get $f(x) = a$, i.e., f becomes a constant function. Therefore we assumed $b \neq ad$ or $c \neq ae$.

It is easy to see that for (2, 2)-rational function (3.1) the equation $f(x) = x$ for fixed points is equivalent to the equation

$$x^3 + (d - a)x^2 + (e - b)x - c = 0. \tag{3.2}$$

Since $\mathbb{C}_p$ is algebraic closed the Eq. (3.2) may have three solutions with one of the following relations:

(i) One solution having multiplicity three;
(ii) Two solutions, one of which has multiplicity two;
(iii) Three distinct solutions.