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Continuum kinematical modeling of mass increasing biological growth

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ABSTRACT

Constructing continuum kinematical models of mass increasing biological growth has been the objective of many studies in the last 100 years. The significant features of the successful studies are briefly described, critically reviewed and organized in the contemporary notation of continuum kinematics. While success has been achieved in many kinematic tissue modeling categories, an agreed upon kinematic model for the large deformations of soft tissue remains a challenge.

1. Introduction

A problem in the continuum kinematic modeling of the deformations of growing objects is that material is added as well as stretched during the growth process. The addition of material means that the mapping will not, in general, be one-to-one and onto. Classical continuum kinematic modeling assumes that the motion of an object is one-to-one and onto, a bijective mapping. In order to continuum kinematically model the deformations of growing objects it is necessary to extend/modify the fundamental continuum kinematical hypothesis of bijective motions. The extension/modification objective of the fundamental continuum kinematical hypothesis is also a change from a closed system approach to an open system approach to modeling biological growth. The difference between open and closed systems is that for a closed system no mass is exchanged between the growing object and its exterior. This means that there is no linear momentum, angular momentum, energy of entropy exchanged between the growing object and its exterior. An open system may exchange mass and each of the quantities just mentioned with its exterior. The open system approach is necessary for modeling interstitial mass increasing biological growth.

The extension of the fundamental continuum kinematical hypothesis actually requires also an extension of the continuum mechanics definition of a body (object) to encompass a growing body (object). (The use of the word "body" in classical mechanics, as in "free body diagram", and "a potato shaped body" is replaced here by the word "object" in order to avoid prose in which the classical mechanics use of the word and the biological use of the word appear together, as in "the free body diagram of a body.") A widely accepted definition of an object (i.e., body) in classical mechanics was given by Noll [19]. This definition (Section 10) involves the specification of the mappings that are one-to-one and onto between different configurations of the object. The extension of the definition of an object to a growing object is considered in Section 10.

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Growth is the process of gradual increase in the net volume of a tissue, but it may also include some resorption. A distinction is made between *appositional* growth and *interstitial* growth; that is the difference between growth on the surface and growth within a volume of tissue. Appositional growth occurs, for example, when a tree adds another layer to the outside of its trunk and interstitial growth occurs when growth occurs within the interstitial cavities or spaces in the tissues of animals or plants. In addition to trees and some plants, hard tissues (bone, teeth, horns, fingernails) grow by apposition and soft tissues grow interstitially. There is evidence that growth in length of an organism is proportional to the cube root of growth in volume. Tissue growth is partially controlled by the recent history of mechanical/chemical loading although growth rate may also be slowed or accelerated by hormones, vitamins, bioelectrical factors, surgical intervention, imbalances between deposition and resorption and other factors. In addition to, and apart from, growth, a mature tissue may remodel its extracellular matrix to adapt its structure to the mechanical/chemical loading environment it is and has been experiencing. Remodeling of this type is not considered in the present analysis, although remodeling is known to occur with some growth processes. Morphogenesis refers to the processes that are responsible for producing the complex shapes of adults from the simple ball of cells that derives from division of the fertilized egg. Morphogenetic events include pattern and template formation in tissue development. Morphogenetic processes interact with the growth and remodeling processes and often all three process types occur simultaneously.

In this contribution the focus is on the kinematics of mass increasing biological growth; neither morphogenesis nor tissue remodeling nor mass loss of tissue is considered. Narrowing the objective sharpens the focus. If no transport occurs, then the mass of the object is constant and it is possible to describe the kinematics of motion as bijective, one-to-one and onto and the system as a closed system. If an object is a closed system, it always retains the same particles. However if the system is open, particles may enter or leave the system and the motion is only bijective under limited conditions. Considerations are restricted here to the possibility of particles entering the system.

In the next section the definition of the motion of an object is introduced. In the section that follows, the definitions of bijective, injective and surjective motions are introduced. Bijective motions characterize closed systems and injective motions offer the possibility of monotonically increasing mass in the continuum model of the motion of an object. Following sections concern bijective motions, injective motions, infinitesimal motions, mass conservation for bijective and injective motions, growth velocity field representations and applications of mixture theory. The extension of the definition of an object to a growing object is considered in the next to last section. The conclusion is a summary and discussion.

2. Motions

In traditional continuum mechanics [33] two coordinate systems are employed for finite deformations. In a reference coordinate system the points in the image of an object O are placed in a one-to-one correspondence with the coordinates of their image. These image coordinates, called particles, are represented by their position vector $\mathbf{X}$ at a reference time t_r, all $\mathbf{x} \subset O(t_r)$ as a complete representation of the object in the reference configuration. The image coordinates of the same particle at time t, called the place of the particle at time t, are represented by their position vector $\mathbf{x}$; all $\mathbf{x} \subset O(t)$ is a complete representation of the object in the instantaneous configuration. The coordinate system for $\mathbf{x}$ is generally different from that for $\mathbf{X}$, but they could be the same. The motion of the particle $\mathbf{X}$ is then given by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \subset O(t_r) \tag{1}$$

which is a set of three scalar-valued functions whose arguments are the particle $\mathbf{X}$ and time t and whose values are the components of the place $\mathbf{x}$ at time t of the particle $\mathbf{X}$. This is the referential description of motion described by Truesdell [34]; see also [16, p. 138]. Since $\mathbf{X}$ can be any particle in the object, $\mathbf{x} \subset O(t_r)$, the motion (1) describes the motion of the entire object $\mathbf{x} \subset O(t)$. Eq. (1) is thus referred to as the *motion* of the object O. This is called the *material description of motion* because the material particles $\mathbf{X}$ are the independent variables. A *deformation* is a motion between two fixed times, t_r and t^*, and is often written as

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t^*) \quad \text{for all } \mathbf{X} \subset O(t_r), \tag{2}$$

perhaps without the t^*.

3. Bijective, injective and surjective motions

A bijective motion is defined more sharply below, but basically it means that the mapping is one-to-one and onto. Thus for every $\mathbf{X}$ there exists an $\mathbf{x}$ and vice versa for all t. The definitions presented of bijective, injective and surjective motions are extensions of the definitions of bijective, injective and surjective functions or mappings [3]. In situations where growth occurs by the addition of mass, and without the loss of mass, the motion is no longer a bijective motion and it could be considered as an injective motion, a term defined below. The consequences of loss of bijectivity on the motion are described in the following sub-section.

A motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ is said to be *injective* (one to one) if for all $\mathbf{X} \subset O(t_r)$, $\boldsymbol{\chi}(\mathbf{X}, t) = \boldsymbol{\chi}(\mathbf{X}^*, t) \Rightarrow \mathbf{X} = \mathbf{X}^*$ for all t or, equivalently, for all $\mathbf{X} \subset O(t_r)$, $\mathbf{X} \neq \mathbf{X}^* \Rightarrow \boldsymbol{\chi}(\mathbf{X}, t) \neq \boldsymbol{\chi}(\mathbf{X}^*, t)$ for all t. Alternatively one could say that an injection maps elements of $O(t_r)$ to at most one element of $O(t)$; not every element of $O(t)$, however, need to have an argument mapped to it.