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Spatial analysis of cities using Renyi entropy and fractal parameters

Yanguang Chen, Jian Feng*

Department of Geography, College of Urban and Environmental Sciences, Peking University, Beijing 100871, PR China

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ABSTRACT

The spatial distributions of cities fall into two groups: one is the simple distribution with characteristic scale (e.g. exponential distribution), and the other is the complex distribution without characteristic scale (e.g. power-law distribution). The latter belongs to scale-free distributions, which can be modeled with fractal geometry. However, fractal dimension is not suitable for the former distribution. In contrast, spatial entropy can be used to measure any types of urban distributions. This paper is devoted to generalizing multifractal parameters by means of dual relation between Euclidean and fractal geometries. The main method is mathematical derivation and empirical analysis, and the theoretical foundation is the discovery that the normalized fractal dimension is equal to the normalized entropy. Based on this finding, a set of useful spatial indexes termed "generalized multifractal indicators" are defined for geographical analysis. These indexes can be employed to describe both the simple distributions and complex distributions. The generalized multifractal indexes are applied to the population density distribution of Hangzhou city, China. The calculation results reveal the feature of spatio-temporal evolution of Hangzhou's urban morphology. This study indicates that fractal dimension and spatial entropy can be combined to produce a new methodology for spatial analysis of city development.

1. Introduction

Cities and systems of cities are complex systems, but a complex system has its simple aspects. In order to understand city development or urban evolution, we must explore the spatial distribution (e.g., urban density distribution) and the related probability distribution (e.g., rank-size distribution) of cities. The spatial distributions and probability distributions can be divided into two types: one is the simple distribution with characteristic scales such as exponential distribution, and the other is complex distribution without characteristic scales such as power-law distribution [3,13,47]. The complex distribution without characteristic scales can be termed scale-free distribution or scaling distribution. One of powerful tools for scaling analysis is fractal geometry [36]. Fractal theory has been applied to urban studies for a long time [8,28,41]. The fractal city studies lead to new urban theory. However, the target of complexity science is not for complexity itself, but for the inherent relationships between complex phenomena and simple rules. Therefore, in urban geographical studies, we should explore both the simple and complex aspects and the connection between spatial complexity and simplicity. The limitation of monofractal method is that it is not suitable for simple distributions. The dimension of a non-fractal distribution is a Euclidean dimension and provides us with no geographical spatial information.

However, it is possible to generalize multifractal measures to describe the spatial distributions with characteristic scale. The new indexes are not real multifractal parameters, but bear analogy with multifractal parameters. The main reasons are as follows. First, multifractal parameters are based on entropy functions, and entropy can be employed to measure both fractal and non-fractal distributions. Second, multifractal measures take on several parameter spectrums, and the spectrums compose both fractal parameters and non-fractal parameters. Third, multifractal dimensions are mainly generalized fractal dimension, and Euclidean dimension can be treated as special cases of fractal dimension. If we find a parameter link between simple distributions and complex distributions, we will be able to generalize multifractal theory and apply it to varied spatial distributions. The link rests with the concept of normalized entropy. In fact, all the spatial distribution data can be converted into probability distribution data, and based on a probability distribution, entropy can be evaluated. Entropy is a measure of complexity [4,5,23]. Hausdorff dimension proved to be equivalent to Shannon entropy and Kolmogorov complexity [43]. The breakthrough point of developing fractal measures is entropy functions and the association of entropy with fractal dimension. A recent finding is that the normalized fractal dimension is theoretically equal to the normalized entropy under certain conditions [11,15]. Thus, the key to solving the problem is to establish a

* Corresponding author.
E-mail addresses: chenyg@pku.edu.cn, fengjian@pku.edu.cn (J. Feng).

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methodological framework for measurements of entropy and fractal dimension.

The methodological framework can be defined by analogy with the box-counting method in fractal theory. Thus, spatial entropy can be naturally associated with fractal dimension. The concept and measurement of spatial Shannon entropy has been introduced to geographical analysis for many years [6,7]. More general spatial entropy is the spatial Renyi entropy [18,24]. Both Renyi entropy and fractal dimension can be employed to measure urban sprawl [39,48], and the similar functions suggest the intrinsic relation between entropy and fractal dimension. The traditional spatial entropy is quantified by means of geographical systems of zones. A zonal system always takes on an irregular network. Based on the irregular nets, entropy cannot be converted into fractal dimension. However, if we use regular grid to replace the irregular network, we will be able to calculate both spatial entropy and fractal dimension. The functional box-counting method is based on the recursive process of regular grid. This method is proposed by Lovejoy et al. [35] and consolidated by Chen [10], and can be applied to both entropy and fractal dimension measurements. Based on the functional box-counting method, the normalized fractal dimension proved to equal the normalized entropy [15]. Based on this equivalence relationship, we can extend the multifractals parameters and obtain useful spatial indexes. This paper is devoted to developing the methodology of spatial analysis using spatial entropy and fractal measures. The functional boxes will be replaced by systems of concentric rings. The rest parts are organized as follows. In Section 2, based on the relationships between spatial Renyi entropy and general correlation dimension, the multifractal measures are generalized to yield a set of new spatial measurements for urban studies; In Section 3, the generalized multifractal parameters are applied to the city of Hangzhou, China, to make a case study; In Section 4, the related questions are discussed, and finally the work is concluded by summarizing the main points. The methodology developed in this article may be used to characterize the spatial structure of other natural and social systems.

2. Models

2.1. Spatial Renyi entropy

It is necessary to clarify the internal relation between spatial entropy and fractal dimension of cities. Both entropy and fractal dimension can serve for the space-filling measures of city development. A central region of a city has higher fractal dimension values (Feng and Chen, 2010), and accordingly, its entropy value is higher than the periphery region [24]. The common fractal dimension formulae are all based on entropy functions. The generalized correlation dimension of multifractals is defined on the base of Renyi's entropy [42], which is formulated as follows

$$M_q = -\frac{1}{q-1}\ln\sum_{i=1}^{N} P_i{}^q, \tag{1}$$

where q denotes the moment order, M_q refers to Renyi's entropy of order q, P_i refers to the occurrence probability of the ith fractal copy, and N to the number of fractal copies ($i=1, 2, 3,\ldots, N$). A fractal copy can be treated as a fractal unit in a fractal set. Thus the generalized correlation dimension can be given as [25,31,37,49]

$$D_q = -\frac{M_q(\varepsilon)}{\ln\varepsilon} = \frac{1}{q-1}\frac{\ln\sum_{i=1}^{N(\varepsilon)} P_i(\varepsilon)^q}{\ln\varepsilon}, \tag{2}$$

in which ε denotes the linear size of fractal copies at given level, and $N(\varepsilon)$ and $P_i(\varepsilon)$ refers to the corresponding fractal copy number and occurrence probability. If we use a box-counting method to make spatial measurements, then ε represents the linear size of boxes, $N(\varepsilon)$ refers to the number of nonempty boxes, and $P_i(\varepsilon)$ to the proportion of geometric objects in the ith box. Based on functional box-counting method, the normalized fractal dimension $D_q{}^*$ proved to be equal to the normalized entropy $M_q{}^*$ [11,15], that is

$$D_q^* = \frac{D_q - D_{\min}}{D_{\max} - D_{\min}} = \frac{M_q - M_{\min}}{M_{\max} - M_{\min}} = M_q^*, \tag{3}$$

where $D_{\max}$ refers to the maximum fractal dimension, $D_{\min}$ to the minimum fractal dimension, $M_{\max}$ refers to the maximum entropy, and $M_{\min}$ to the minimum entropy. In theory, for the fractal cities defined in a 2-dimensional embedding space [8], the basic parameters are as follows: $D_{\max}=2$, $D_{\min}=0$, $M_{\max}=\ln N_T$, $M_{\min}=0$, and N_T is total number of all possible fractal units or boxes (nonempty boxes and empty boxes) in a study area. Thus, Eq. (3) can be reduced to

$$D_q^* = \frac{D_q}{D_{\max}} = \frac{M_q}{M_{\max}} = M_q^*, \tag{4}$$

which is valid only for fractal systems. This suggests that the ratio of the actual fractal dimension to the maximum fractal dimension is theoretically equal to the ratio of the actual entropy to the maximum entropy of a fractal object. The relation between entropy and fractal dimension is supported by the observational data of cities such as Beijing and Hangzhou [15].

2.2. Generalized multifractal parameters

Fractal dimension can be only applied to fractal objectives, and we cannot use fractal geometry to model the non-fractal phenomena. In fact, a non-fractal system can be described with the common measures such as length, area, volume, and density rather than fractal dimension. In other words, the standard fractal parameters have strict sphere of application: irregular and self-similar patterns, nonlinear and recursive process, and complex and scale-free distributions [10]. In contrast, entropy can be utilized to measure any distributions of cities, including fractals and non-fractals. However, for the fractal distributions, entropy values depend on spatial scales of measurements. Fortunately, using Eq. (4), we can generalize fractal measurements, and define a set of quasi-fractal parameters for spatial analysis. In fact, for non-fractals, we have $M_q{}^*=M_q/M_{\max}$, but we have no $D_q{}^*=D_q/D_{\max}$ because D_q is not existent. Now, we can define a general normalized fractal parameter as follows

$$D_q^* = M_q^* = \frac{M_q}{M_{\max}}, \tag{5}$$

in which M_q and $M_{\max}$ are measurable. For a non-fractal object, the maximum dimension is just the Euclidean dimension of the embedding space, that is $D_{\max}=d$. Thus we can further define an apparent multifractal dimension as below:

$$D_q = D_{\max}D_q^* = dM_q^* = \frac{2M_q}{M_{\max}}, \tag{6}$$

which can be theoretically demonstrated [15]. The word "apparent" means "(a thing) that seems to be real or true but may not be", and apparent multifractal measures are not real multifractal measures. Actually, the word of apparent is adopted here to avoid conflicts with existing term of "generalized correlation dimension" in the literature. If a city is examined in a 2-dimensional space, then $D_{\max}=d=2$. In fractal theory, the mass exponent can be given by

$$\tau_q = (q-1)D_q, \tag{7}$$

in which τ_q denotes the common mass exponent. Accordingly, we can define a generalized mass exponent such as

$$\tau_q = (q-1)dM_q^* = \frac{2(q-1)M_q}{M_{\max}}, \tag{8}$$