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On deforming a sector of a circular cylindrical tube into an intact tube: Existence, uniqueness, and stability

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A B S T R A C T

Within the context of finite deformation elasticity theory the problem of deforming an open sector of a thick-walled circular cylindrical tube into a complete circular cylindrical tube is analyzed. The analysis provides a means of estimating the radial and circumferential residual stress present in an intact tube, which is a problem of particular concern in dealing with the mechanical response of arteries. The initial sector is assumed to be unstressed and the stress distribution resulting from the closure of the sector is then calculated in the absence of loads on the cylindrical surfaces. Conditions on the form of the elastic strain-energy function required for existence and uniqueness of the deformed configuration are then examined. Finally, stability of the resulting finite deformation is analyzed using the theory of incremental deformations superimposed on the finite deformation, implemented in terms of the Stroh formulation. The main results are that convexity of the strain energy as a function of a certain deformation variable ensures existence and uniqueness of the residually-stressed intact tube, and that bifurcation can occur in the closing of thick, widely opened sectors, depending on the values of geometrical and physical parameters. The results are illustrated for particular choices of these parameters, based on data available in the biomechanics literature.

1. Introduction

Residual stresses have a very important role in the mechanical functioning of arteries and the existence of residual stresses is well documented, as in, for example, Chuong and Fung [1], Vaishnav and Vossoughi [25] and Fung [5]; see also the review by Humphrey [15]. They are demonstrated by the simple device of cutting radially a short ring of an artery, the result of the cut being the springing open of the ring into a sector, thereby releasing some residual stresses; in general, some residual stress remains, however, as shown by Vossoughi et al. [26] and Greenwald et al. [7]; see also the review by Rachev and Greenwald [21]. A crude estimate of the circumferential residual stresses is obtained by measuring the resulting angle of the sector into which the ring deforms, although it has to be said that reliable quantitative data remain elusive. This so-called *opening angle method* has been analyzed in some detail by several authors, including, for example, Delfino et al. [2], Zidi et al. [27], Rachev and Hayashi [22], Holzapfel et al. [13], Ogden and Schulze-Bauer [19], Matsumoto and Sato [16], Ogden [18], Raghavan et al. [23] and Olsson et al. [20]. The typical approach is to assume that the (unloaded) opened sector is free

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of stress and circular cylindrical and then to construct a deformation that brings the sector into an intact tube and to calculate the resulting radial and circumferential (residual) stress distributions. This has been done for a single layer and for two-layered tubes. In general this method does not account for any stress or deformation in the axial direction. However, in reality, not only does the ring open into a sector, but the length of the arterial segment tends to change and there are also residual stresses in the axial direction, although quantitative information about axial residual stresses is distinctly lacking. A recent 3D analysis by Holzapfel and Ogden [14] is the first attempt to calculate the combination of radial, circumferential and axial residual stresses. The purpose of the present paper is to provide an analysis of the opening angle method, with particular reference to the questions of existence and uniqueness of the residually-stressed configuration and its stability.

In Section 2 we review the description of the geometry of the deformation which first takes the sector into an intact cylinder, allowing for a possible length change, and then allows inflation under internal pressure combined with axial load while maintaining circular cylindrical symmetry. The corresponding stress components and equilibrium equations are then summarized in respect of an isotropic elastic material whose properties are described in terms of a strain-energy function. The existence and uniqueness of the resulting configuration is then examined in Section 3, under the assumption that the strain energy is convex as a function of a suitably chosen deformation variable. The Mooney–Rivlin and the Fung strain energies obey this assumption, and we use them, as well as some experimental data on arteries, to illustrate the analysis. Section 4 is devoted to an analysis of the stability of the residually stressed tube (in the absence of lateral loads) on the basis of the theory of small incremental deformations superimposed on a finite deformation, and for this purpose an appropriate version of the Stroh formulation is adopted. For simplicity of illustration, attention is restricted to prismatic incremental deformations. We then treat numerically the case of tubes made of solids with the Mooney–Rivlin strain energy, and compare the predictions to some simple experiments we performed with silicone rubber.

2. Problem formulation

2.1. Geometry of the deformation

We consider an annular sector of a circular cylindrical tube with geometry defined by

$$A \leqslant R \leqslant B, \quad -(2\pi - \alpha)/2 \leqslant \Theta \leqslant +(2\pi - \alpha)/2, \quad 0 \leqslant Z \leqslant L, \tag{1}$$

where A and B are the radii of the inner and outer faces, respectively, of the sector, L is its length, and (R,Θ,Z) denote the cylindrical coordinates of a point in the material, with corresponding orthonormal basis vectors $(\boldsymbol{E}_R,\boldsymbol{E}_\Theta,\boldsymbol{E}_Z)$. Here $\alpha \in (0,2\pi)$ is the so-called *opening angle*. The sector is assumed to be unstressed in this reference configuration, which we denote by $\mathcal{B}_0$.

The sector is then deformed into an intact (circular cylindrical) tube so that the plane faces originally at $\Theta = \pm(\pi - \alpha/2)$ are joined perfectly together and there is an accompanying uniform axial stretch $\lambda_z = l/L > 1$, where l is the deformed length of the tube. We refer to this new configuration as the *residually-stressed* configuration, which we denote by $\mathcal{B}_r$. In this configuration there is no traction on the curved surfaces of the tube, but in general axial loads are required to maintain the deformation since it turns out that the axial stress depends on the radius r and cannot therefore be set to zero pointwise on the ends of the tube. Finally, the tube is subjected to a uniform internal pressure of magnitude P per unit deformed area and is maintained in a new circular cylindrical configuration, which we refer to as the *loaded configuration*, denoted $\mathcal{B}_l$; the three separate configurations are depicted in Fig. 1.

Let (r,θ,z) denote cylindrical coordinates in either the residually-stressed configuration $\mathcal{B}_r$ or the pressurized configuration $\mathcal{B}_l$, with corresponding orthonormal basis vectors $(\boldsymbol{e}_r,\boldsymbol{e}_\theta,\boldsymbol{e}_z)$. The deformation may be described by the equations

$$r = r(R), \quad \theta = k\Theta, \quad z = \lambda_z Z, \tag{2}$$

from which we may calculate the deformation gradient tensor $\boldsymbol{F} = \mathrm{Grad}\,\boldsymbol{x}$ as

$$\boldsymbol{F} = r'\boldsymbol{e}_r \otimes \boldsymbol{E}_R + (kr/R)\boldsymbol{e}_\theta \otimes \boldsymbol{E}_\Theta + \lambda_z \boldsymbol{e}_z \otimes \boldsymbol{E}_Z, \tag{3}$$

where the prime denotes differentiation and

$$k \equiv \frac{2\pi}{2\pi - \alpha}, \quad k > 1 \tag{4}$$

is *a measure of the opening angle*. The left Cauchy–Green tensor is then calculated as

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = r'^2\boldsymbol{e}_r \otimes \boldsymbol{e}_r + (kr/R)^2\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \lambda_z^2\boldsymbol{e}_z \otimes \boldsymbol{e}_z, \tag{5}$$

and it follows immediately that the principal axes of $\boldsymbol{B}$ are $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_z$. The corresponding principal stretches are

$$\lambda_1 = r', \quad \lambda_2 = \frac{kr}{R}, \quad \lambda_3 = \lambda_z. \tag{6}$$

Arterial wall tissue is generally considered to be essentially *incompressible*, so that the constraint $\det\boldsymbol{F} = 1$ is enforced. For the present deformation this constraint yields

$$r^2 = \frac{R^2 - A^2}{k\lambda_z} + a^2, \tag{7}$$