Chaos, Solitons and Fractals 91 (2016) 25–32

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Review

Multifractal signal reconstruction based on singularity power spectrum

Gang Xiong [a,*], Wenxian Yu [a], Wenxiang Xia [b,*], Shuning Zhang [a]

[a] Shanghai Key Laboratory of Intelligent Sensing and Recognition, Shanghai Jiao Tong University, Shanghai, 200240, P. R. China
[b] Simulation and training center, Logistics Academy, 100858, Beijing, P. R. China

ARTICLE INFO

Article history:
Received 23 November 2015
Revised 23 March 2016
Accepted 29 April 2016

Keywords:
Multifractal reconstruction
Singularity exponent
Singularity power spectrum
Multifractal spectrum

ABSTRACT

Fractal reconstruction (FR) and multifractal reconstruction (MFR) can be considered as the inverse problem of singularity spectrum analysis, and it is challenging to reconstruct fractal signal in accord with multifractal spectrum (MFS). Due to the multiple solutions of fractal reconstruction, the traditional methods of FR/MFR, such as FBM based method, wavelet based method, random wavelet series, fail to reconstruct fractal signal deterministically, and besides, those methods neglect the power spectral distribution in the singular domain. In this paper, we propose a novel MFR method based singularity power spectrum (SPS). Supposing the consistent uniform covering of multifractal measurement, we control the traditional power law of each scale of wavelet coefficients based on the instantaneous singularity exponents (ISE) or MFS, simultaneously control the singularity power law based on the SPS, and deduce the principle and algorithm of MFR based on SPS. Reconstruction simulation and error analysis of estimated ISE, MFS and SPS show the effectiveness and the improvement of the proposed methods compared to those obtained by the Fraclab package.

1. Introduction

Fractal theory provides a powerful tool to model and process ubiquitous self-similar, scale- invariant and non-stationary signal in nature [1–3]. The signal analysis and processing methods based on mono-fractal and multifractal model are widely used as basis research in signal [4–6] and image processing [7,8], biomedicine, geology and atmospheric remote sensing, etc. [9–11]. The multifractal reconstruction includes two situations, and the first is about multifractal modeling and fractal parameter estimation for specific time series and the observed variables of system state; There are abundant research fruits about fractal/multifractal modeling and fractal parameter estimation, such as WTMM [12,13], MF-DFA [14–18], MF-DMA [23,24], wavelet leaders[25], which have been applied in the multifractal modeling and analysis of Internet data flow, radar sea clutter and natural scene [26,27]. The second is about fractal signal reconstruction from given multifractal spectrum and other singularity spectrum, to estimate and predict the nonlinear system state. In this paper, we will focus on the second situation.

Fractal reconstruction, proposed with the emergence of fractal theory, has a long research history [28–32]. The fractal reconstruction of specific dimension has important significance in understanding fractals. Turcotte D. L. constructed cantor ternary sets, and achieved arbitrary dimension cantor structure [33]. Based on IFS, Barnsley M. F. proposed the fractal interpolation function to reconstruct the fractal signal [34]. Zhou studied the regular fractal reconstruction with strict self-similar construction [35]. Riedilf H. Riedi constructed the multifractal processes by using iterative multiply multiplication of fractal Brownian motion (FBM), self-similar processes, Weirstrass process, or Levy process [10]. However, the above methods, restricted by the mono-fractal functions, present non-universality. Furthermore, the above methods with ideal mathematical structure cannot meet the demand of fractal signal reconstruction in many engineering fields, hence, multifractal reconstruction methods in the statistical sense were proposed.

At present, the exiting multifractal reconstruction methods are based on multifractal spectrum, such as Chan-Wood method [36], wavelet transform model maximum (WTMM) method [13], multifractal detrended fluctuation analysis (MFDFA) method [18]. There are also many other methods to construct multifractal signals, many of which have preset multifractal spectra, such as the P-model [19], multifractality in asset returns(MMAR) model [20,21], multifractal random walk [22], and so on. Due to the multiplicity of fractal reconstruction, the above methods have at least two disadvantages, (1) only considering singularity exponent and neglecting the singularity power spectrum, the reconstructed signal is arbitrary and uncontrollable about power spectrum in the singularity domain. For example, the SPS of the reconstructed

* Corresponding authors.
E-mail addresses: gxiong@sjtu.edu.cn (G. Xiong), wxyu@sjtu.edu.cn (W. Yu), pla56boshi@126.com (W. Xia), shuningzhang0704@163.com (S. Zhang).

http://dx.doi.org/10.1016/j.chaos.2016.04.021

signal based on WTMM method depends on the Gaussian wavelet coefficients at the large scales. (2) The reconstructed signals have no uniqueness in the power distribution in the singularity domain, i.e., two signals with same multifractal spectrum may have entirely different SPS, which cause the confusion in the recognition and classification. Xiong et al. [37] introduced a novel fractal energy measurement and the singularity energy spectrum, and Xiong et al. [38] proposed the singularity power spectrum (SPS) and proved that two multifractal signals with similar multifractal spectra may have diverse SPS. These researches provide new idea for multifractal reconstruction (MFR). In this paper, we propose a novel multifractal reconstruction (MFR) based on SPS distribution (MFR-SPS).

The rest of the paper is organized as follows. The next section, Section 2, presents preliminaries on fractal reconstruction, including conception of SPS and MFR-SPS, fundamental hypothesis and feasibility analysis. Definitions and algorithms of multifractal reconstruction based on SPS and multifractal spectrum (MFS), or SPS and instantaneous singularity exponents (ISE) are studied in Section 3. The reconstructing process of several typical multifractal signals can be found in Section 4. Furthermore, the singularity properties, including multifractal spectrum, SPS and instantaneous singularity exponents are analyzed to validate the availability of proposed method. Finally, the results can be found in Section 5.

2. Feasibility of SPS based fractal reconstruction

2.1. Principle problem and hypothesis

Except for the 3-divided Cantor set and regular self-similar fractal models, the multiplicity of MFR makes it impossible to obtain the determining reconstruction, respectively based on fractal dimension, multifractal spectrum (MFS), instantaneous singularity exponent (ISE), or singularity power spectrum (SPS) distribution. There are two possible solutions of eliminating the multiplicity, i.e., reasonable hypothesis of singularity exponent distribution and multi-dimensional joint controlling of spectrum distribution in the singularity domain. We will give the hypothesis about the distribution of the ISE and explore novel reconstruction method based on the SPS and ISE or MFS.

Representation of fractal reconstruction. Given instantaneous singularity exponent $\alpha(t)$, multifractal spectrum $f(\alpha)$ and SPS $p_x(\alpha)$, we can reconstruct the singularity subset $x_\alpha(t)$ by some fractal structure, and then reconstruct the whole fractal signal $x(t) = \sum_\alpha x_\alpha(t)$. Note that reconstruction of fractal subset plays a key role, and when we approximate the Hausdorff spectrum with Legendre transform and Legendre spectrum, the consistent uniform covering hypothesis ($N^{(n)}(\alpha, \varepsilon, t)$ counting method) is selected as a special structure to replace the optimal covering of Hausdorff measurement to calculate multifractal spectrum. Before reconstruction of fractal signal, we intend to introduce the consistent uniform covering hypothesis, and suppose that:

(1) *Zero mean characteristics* of fractal signal, which guarantees the multifractal subsets have fully developed and have the power uniform distribution characteristics.
(2) *Stationary of reconstructed wavelet coefficients* of fractal signal, which makes it possible to obtain singularity subsets with uniform distribution.
(3) *Consistent uniform covering*, i.e., replacing the optimal coverage with $N^{(n)}(\alpha, \varepsilon, t)$ counting method, which ignores the diversity of the fractal structure and simplifies diverse singular structure of multifractal signal by consistent uniform covering.

2.2. Conceptual model of MFR

Given the SPS, multifractal spectrum (MFS) or instantaneous singularity exponent (ISE), the multifractal signal reconstruction can be modeled as shown in Fig. 1. SPS function represents the power measurement of singular subsets, which shows the power distribution in the singularity domain, while MFS or ISE describes the statistical distribution of singularity exponents and fractal dimension of singular subsets, and also originates from the differentiability and relationship between single points and overall signal. The reconstructed signal only based on SPS cannot determinate the distribution of singularity exponents, and thus fails to reconstruct the fractal signal exclusively. On the contrary, the reconstructed signal only based on MFS or ISE ignores power distribution of singular subsets, and thus cannot reconstruct accurately the fractal signal.

With this viewpoint, we demonstrate the reconstruction modeling, as shown in Fig. 1, i.e., multifractal reconstruction methods based on SPS and MFS, or based on SPS and ISE. Note that MFS can be obtained with statistical calculation of ISE, and inversely we can deduce statistical distribution of ISE with MFS. Based on the singularity power controller and ISE controller, we can generate the statistical wavelet coefficients conform to the multifractal characteristics defined by SPS and ISE or MFS, and then by the multiscale wavelet inverse transform, the multifractal time series can be reconstructed.

3. MFR based on SPS

3.1. Singularity power spectrum (SPS)

Combining the traditional power spectrum analysis and multifractal spectrum analysis, the singularity power spectrum distribution explains the signal power distribution based on the variation of singularity exponents, and provides a new fractal signal analysis and processing method.

3.1.1. Singularity Energy spectrum [37]

For multifractal signal $x(t)$, suppose that $x(t)$ can be expressed as a union set of dense singularity subsets $x_\alpha(t)$, which is also known as fractal sub-band signals or subsets, i.e. $x(t) = \cup_\alpha x_\alpha(t)$, and then singularity energy spectrum of $x(t)$ can be expressed as [37]

$$E_x(\alpha) = \int_{-\infty}^{\infty} \frac{|x_\alpha(t)|^2}{\sqrt{1+\tan^2\theta_t}} H(x_\alpha(dt)), \tag{1}$$

where $H(x_\alpha(dt))$ is the differential of Hausdorff measurement of fractal subsets, and θ_t is local direction angle. For finite energy fractal signal, singularity energy spectrum represents signal energy distribution of different singularity fractal subsets or sub-band signal.

Researches show that singularity energy spectrum expresses the energy measure of the multifractal signal in the hierarchy of singularity exponent, which describes the contribution of energy of different singularity exponents, and the distribution of fractal energy in different scales and hierarchy [37,38].

3.1.2. Singularity power spectrum

the SPS function of $x(t)$ can be defined as the limit form [38]

$$p_x(\alpha) = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|x_\alpha^T(t)|^2}{T(x_\alpha^T(t))\sqrt{1+\tan^2\theta_t}} H(x_\alpha(dt)), \tag{2}$$

where $x_\alpha^T(t)$ denotes the truncation version of $x_\alpha(t)$, $T(x_\alpha^T(t))$ is fractal truncation interval, θ_t and $H(x_\alpha(dt))$ are the same with Eq. (1).