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A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions

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ABSTRACT

We study a nonlocal boundary value problem of Hadamard type coupled sequential fractional differential equations supplemented with coupled strip conditions (nonlocal Riemann-Liouville integral boundary conditions). The nonlinearities in the coupled system of equations depend on the unknown functions as well as their lower order fractional derivatives. We apply Leray-Schauder alternative and Banach's contraction mapping principle to obtain the existence and uniqueness results for the given problem. An illustrative example is also discussed.

1. Introduction

Fractional calculus has emerged as an interesting field of investigation in the last few decades. The subject has been extensively developed and the literature on the topic is much enriched now, covering theoretical as well as widespread applications of this branch of mathematical analysis. In fact, the tools of fractional calculus have led to much improved, realistic and practical mathematical modeling of many systems and processes, occurring in engineering and scientific disciplines such as control theory, signal and image processing, biophysics, blood flow phenomena, etc. [1–3]. The nonlocal characteristic of fractional order operators, which takes into account the hereditary properties of phenomena involved, has greatly contributed to the popularity of the subject. In particular, the topic of boundary value problems of Riemann-Liouville or Liouville-Caputo type fractional order differential equations equipped with a variety of boundary conditions has attracted significant attention, for example, see [4–14] and the references cited therein.

In the classical text [15], it has been mentioned that Hadamard in 1892 [16] suggested a concept of fractional integro-differentiation in terms of the fractional power of the type $(t\frac{d}{dt})^q$ in contrast to its Riemann-Liouville counterpart of the form $(\frac{d}{dt})^q$. The kind of derivative introduced by Hadamard contains logarithmic function of arbitrary exponent in the kernel of the integral appearing in its definition. Hadamard's construction is invariant in relation to dilation and is well suited to the problems containing half axes. In [18], the authors have shown that the Lamb-Bateman integral equation can be expressed in terms of Hadamard fractional derivatives of order 1/2. Following an approach based on Hadamard derivatives and fractional Hyper-Bessel-type operators, the authors in [17] discussed a modified Lamb-Bateman equation. In fact, there are not many applications of Hadamard integrals and derivatives in view of possibly the intrinsic intractability of these operators. However, one can find some recent results on Hadamard type fractional differential equations, for instance, see [2,19–23].

Coupled systems of fractional order differential equations have also been investigated by many authors. Such systems appear naturally in many real world situations, for example, see [24]. Some recent results on the topic can be found in a series of

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papers [25–30] and the references cited therein. More recently, in [27], the authors discussed a coupled system of Hadamard type fractional differential equations with Hadamard integral boundary conditions of the form:

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)),\ D^{\beta}v(t) = g(t, u(t), v(t)), \\ \quad 1 < \alpha, \beta \le 2,\ 1 < t < e, \\ u(1)=0,\ u(e) = \frac{1}{\Gamma(\gamma)}\int_1^{\sigma_1}\left(\log\frac{\sigma_1}{s}\right)^{\gamma-1}\frac{u(s)}{s}ds,\ \gamma > 0,\ 1 < \sigma_1 < e, \\ v(1) = 0,\ v(e) = \frac{1}{\Gamma(\gamma)}\int_1^{\sigma_2}\left(\log\frac{\sigma_2}{s}\right)^{\gamma-1}\frac{v(s)}{s}ds,\ 1 < \sigma_2 < e, \end{cases} \tag{1.1}$$

where $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In this paper, motivated by aforementioned applications of Hadamard fractional integro-differentiation as well as by the recent interest in developing the existence theory for Hadamard type initial and boundary value problems, we study a coupled system of Hadamard type sequential fractional differential equations equipped with nonlocal coupled strip conditions given by

$$\begin{cases} (D^q + kD^{q-1})u(t) = f(t, u(t), v(t), D^{\alpha}v(t)),\ k > 0, \\ \quad 1 < q \leqslant 2,\ 0 < \alpha < 1 \\ (D^p + kD^{p-1})v(t) = g(t, u(t), D^{\delta}u(t), v(t)), \\ \quad 1 < p \leqslant 2,\ 0 < \delta < 1, \\ u(1) = 0,\ u(e) = I^{\gamma}v(\eta) = \frac{1}{\Gamma(\gamma)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma-1}\frac{v(s)}{s}ds,\ \gamma > 0, \\ \quad 1 < \eta < e, \\ v(1) = 0,\ v(e) = I^{\beta}u(\zeta) = \frac{1}{\Gamma(\beta)}\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\beta-1}\frac{u(s)}{s}ds,\ \beta > 0, \\ \quad 1 < \zeta < e, \end{cases} \tag{1.2}$$

where $D^{(.)}$ and $I^{(.)}$ respectively denote the Hadamard fractional derivative and Hadamard fractional integral (to be defined later), and $f,\ g : [1, e] \times \mathbb{R}^3 \to \mathbb{R}$ are given continuous functions to be chosen appropriately (according to the requirement).

We show the existence of solutions for problem (1.1) by applying Leray-Schauder alternative criterion while uniqueness of solutions for (1.1) relies on Banach's contraction mapping principle. Though we make use of the standard methodology to obtain the desired results, yet its exposition to the given problem is new. Section 2 contains some basic concepts and an auxiliary lemma, while the main results are presented in Section 3.

Here we remark that problem (1.2) is a generalization of the problem (1.1) in the sense that the coupled system in (1.2) consists of sequential fractional differential equations with the nonlinearities depending on the unknown functions as well as their lower order fractional derivatives, and coupled strip boundary conditions, whereas problem (1.1) deals with fractional Hadamard type equations with nonlinearities only involving unknown functions and uncoupled integral boundary conditions.

2. Preliminaries

First of all, we recall some important definitions [2] and then prove an auxiliary lemma for the linear variant of problem (1.1).

Definition 2.1. The Hadamard fractional integral of order q for a function $g \in L^p[y, x], 0 \le y \le t \le x \le \infty$, is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)}\int_y^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds,\ q > 0.$$

Definition 2.2. Let $[y, x] \subset \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC^n_{\delta}[y, x] = \{g : [y, x] \to \mathbb{R} : \delta^{n-1}[g(t)] \in AC[y, x]\}$. The Hadamard derivative of fractional order q for a function $g \in AC^n_{\delta}[y, x]$ is defined as

$$D^q g(t) = \delta^n(I^{n-q})(t) = \frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-q-1}\frac{g(s)}{s}ds,$$
$$n - 1 < q < n,\ n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Recall that the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral in the space $L^p[y, x]$, $0 < y < x < \infty$, $1 \le p \le \infty$, that is, $D^q I^q f(t) = f(t)$ (Theorem 4.8, [31]).

In [32], Caputo-type modification of the Hadamard fractional derivatives was proposed as follows:

$$^cD^q g(t) = D^q\left[g(s) - \sum_{k=0}^{n-1}\frac{\delta^k g(y)}{k!}\left(\log\frac{s}{y}\right)^k\right](t),\ t \in (y, x). \tag{2.1}$$

Further, it was shown in (Theorem 2.1, [32]) that $^cD^q g(t) = I^{n-q}\delta^n g(t)$. For $0 < q < 1$, it follows from (2.1) that

$$^cD^q g(t) = D^q\Big[g(s) - g(y)\Big](t).$$

Furthermore, it was established in Lemma 2.4 and Lemma 2.5 of [32] respectively that

$$^cD^q(I^q g)(t) = g(t),\ \ I^q(^cD^q)g(t) = g(t) - \sum_{k=0}^{n-1}\frac{\delta^k g(y)}{k!}\left(\log\frac{t}{y}\right)^k. \tag{2.2}$$

From the second formula in (2.2), one can easily infer that the solution of Hadamard differential equation: $^cD^q u(t) = \sigma(t)$ can be written as

$$u(t) = I^q\sigma(t) + \sum_{k=0}^{n-1}\frac{\delta^k u(y)}{k!}\left(\log\frac{t}{y}\right)^k,$$

for appropriate function $u(t)$ and $\sigma(t)$ (as required in the above definitions).

Note that the Hadamard integral and derivative defined above are left-sided. One can define the Hadamard right-sided integral and derivative in the same way, for instance, see [32].

Lemma 2.1. *Let $h_1, h_2 \in AC([1, e], \mathbb{R})$. Then the solution for the linear system of sequential fractional differential equations:*

$$\begin{aligned} (D^q + kD^{q-1})u(t) &= h_1(t), \\ (D^p + kD^{p-1})v(t) &= h_2(t), \end{aligned} \tag{2.3}$$

supplemented with the boundary conditions in (1.2) is given by a coupled system of Hadamard integral equations

$$\begin{aligned} u(t) = \frac{1}{\Delta}&\left(t^{-k}\int_1^t s^{k-1}(\log s)^{q-2}ds\right) \\ &\times\Bigg\{-A_2\Bigg[\frac{e^{-k}}{\Gamma(p-1)}\int_1^e s^{k-1}\left(\int_1^s\left(\log\frac{s}{r}\right)^{p-2}\frac{h_2(r)}{r}dr\right)ds \\ &- \frac{1}{\Gamma(q-1)\Gamma(\beta)}\int_1^{\zeta}\left(\log\frac{\zeta}{s}\right)^{\beta-1}s^{-k-1} \\ &\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{q-2}\frac{h_1(m)}{m}dm\right)dr\right)ds\Bigg] \\ &- B_2\Bigg[\frac{1}{\Gamma(p-1)\Gamma(\gamma)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma-1}s^{-k-1} \\ &\times\left(\int_1^s r^{k-1}\left(\int_1^r\left(\log\frac{r}{m}\right)^{p-2}\frac{h_2(m)}{m}dm\right)dr\right)ds \end{aligned}$$