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Robust Synchronization of Fractional-Order Chaotic Systems at a Pre-Specified Time Using Sliding Mode Controller with Time-Varying Switching Surfaces

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ABSTRACT

A main problem associated with the synchronization of two chaotic systems is that the time in which complete synchronization will occur is not specified. Synchronization time is either infinitely large or is finite but only its upper bound is known and this bound depends on the systems' initial conditions. In this paper we propose a method for synchronizing of two chaotic systems precisely at a time which we want. To this end, time-varying switching surfaces sliding mode control is used and the control law based on Lyapunov stability theorem is derived which is able to synchronize two fractional-order chaotic systems precisely at a pre specified time without concerning about their initial conditions. Moreover, by eliminating the reaching phase in the proposed synchronization scheme, robustness against existence of uncertainties and exogenous disturbances is obtained. Because of the existence of fractional integral of the sign function instead of the sign function in the control equation, the necessity for infinitely fast switching be obviated in this method. To show the effectiveness of the proposed method the illustrative examples under different situations are provided and the simulation results are reported.

1. Introduction

Fractional-order integral and derivative are mathematical notions that their origin are traced back to the end of 17th century. Owing to its application to physics and engineering, recently fractional calculus has attracted increasing attention.

Chaotic systems are nonlinear deterministic systems that display complex and noise-like behavior. These systems are extremely sensitive in respect to initial conditions. Synchronization of chaos has been widely investigated in many fields, such as physics, chemistry, secure communication, power electronic, biological systems [1]. Chaotic behaviors are shown in many fractional-order systems such as Chua [2], Chen [3–5], and Rossler [6], Lorenz [7], Liu [8,9], Lu [10] and Arneodo systems [11].Various control methods have been utilized for synchronizing fractional-order chaotic systems, such as: active control method [12–16] observer-based synchronization [17,18], adaptive control [19,20] and sliding mode control [21–24]. In all that works, due to the existence of the reaching phase, the synchronization techniques are not completely robust and infinitely fast switching are necessary for keeping the systems trajectories on the sliding surfaces. In [25-27]the first problem has been resolved by adopting a reaching law. Gao and Liao in [28] solved this problem by eliminating the reaching phase. The need for infinitely fast switching has been also obviated in [29-31].

On the other hand, in all above mentioned papers, the synchronization time either is infinitely large (asymptotically stability proof of error dynamics have been provided) or is finite but only an upper bound for it has been introduced, which this bound depends on the difference between masters and slaves initial conditions. The more difference in initial conditions, the more time to synchronize chaotic systems.

Motivated by the above discussion, the main contribution of this paper is to synchronize two fractional chaotic systems just at a time when has been set in advance. To achieve this purpose, we have proposed a sliding mode controller with time-varying switching surfaces. For the first time, we have introduced a control law that is able to synchronize two fractional chaotic systems precisely at any time when we want without worrying about how much the two chaotic systems are far from each other at initial time. The control law is derived based on Lyapunov stability technique. Furthermore, the synchronization is made completely robust to uncertainty and disturbances by eliminating the reaching phase. Since the fractional integral of the sign function instead of the sign func-

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tion is used in control equation, the need for infinitely fast switching is also obviated.

2. Preliminaries

In this section, some definitions, and lemmas which are necessary for obtaining a synchronizing controller has been presented.

Definition 1. [32]: Assume $Re\,\alpha > 0$ and assume $f$be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval $[0, \infty)$. Then for $t > 0$ the equality

$$D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \tag{1}$$

is called the Riemann-Liouville α order fractional integral of f.

For simplicity, the term 'fractional integral' is used instead of Riemann-Liouville fractional-order integral. According to definition 1, the fractional integral of t^μ is [1]:

$$D_t^{-\alpha} t^\mu = I_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)} t^{\mu+\alpha}, \quad \alpha > 0,\ t > 0,\ \mu > -1 \tag{2}$$

Definition 2. [33]: The Riemann-Liouville (RL) fractional derivatives of the order $\alpha > 0$, $n-1 < \alpha < n$, $n \in N$, are defined as:

$${}_0^{RL}D_t^\alpha f(t) = \frac{d^n}{dt^n}\left(I_t^{n-\alpha} f(t)\right) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \tag{3}$$

Definition 3. [32]: The Caputo (C) fractional derivative of a function of the order α is:

$${}_0^{C}D_t^\alpha f(t) = \frac{d^n}{dt^n}\left(I_t^{n-\alpha} f(t)\right) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 \le \alpha < n \tag{4}$$

According to the Definition 3, the C fractional derivative of the function $(t)^\beta$ is equal to:

$${}_0^{C}D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad t > 0,\ \beta > -1 \tag{5}$$

The RL and C fractional derivatives of the constant function $\boldsymbol{f(t) = \mu}$ are respectively as follows:

$$\begin{gathered} {}_0^{RL}D_t^\alpha \mu = \frac{\mu}{\Gamma(1-\alpha)} t^{-\alpha}, \\ {}_0^{C}D_t^\alpha \mu = 0, \\ t > 0 \end{gathered} \tag{6}$$

Definition 4. [34]: The incomplete Beta Function, which is a generalization of the Beta function, is defined as follows:

$$B_x(p,q) = \int_0^x u^{p-1}(1-u)^{q-1} du, \quad Re(p) > 0, \quad Re(q) > 0,\ 0 \le x \le 1 \tag{7}$$

Lemma 1. [35]: If $f(t) \in C^m[0, \infty)$ and $0 < \alpha < 1 \in Z^+$, then

$${}_a^{RL}D_t^\alpha\, {}_a^{RL}D_t^{-\alpha} f(t) = {}_a^{C}D_t^\alpha\, {}_a^{RL}D_t^{-\alpha} f(t) = f(t) \tag{8}$$

Lemma 2. [36]: The RL fractional derivative operators commute, i.e.

$${}_a^{RL}D_t^\alpha\left({}_a^{RL}D_t^\beta f(t)\right) = {}_a^{RL}D_t^\beta\left({}_a^{RL}D_t^\alpha f(t)\right) = {}_a^{RL}D_t^{\alpha+\beta} f(t) \tag{9}$$

only if the following conditions are satisfied:

$$f^{(j)}(a) = 0, \quad j = 0, 1, 2, \ldots, r-1 \tag{10}$$

where $m-1 < \alpha < m$, $n-1 < \beta < n$ and $r = \max(n, m)$.

Lemma 3. [35]: If $f(t) \in C^1[0, T]$ for some $T > 0$, then

$${}_0^{C}D_t^\alpha\left({}_0^{C}D_t^\beta f(t)\right) = {}_0^{C}D_t^\beta\left({}_0^{C}D_t^\alpha f(t)\right) = {}_0^{C}D_t^{\alpha+\beta} f(t), \quad t \in [0, T] \tag{11}$$

where $\alpha, \beta \in R^+$ and $\alpha + \beta \le 1$.

Lemma 4. [37]: If $x_i \in R$ for $i = 1, \ldots, n$ and $p \in (0, 1]$, then the following inequality holds:

$$(|x_1| + |x_2| + \cdots + |x_n|)^p \le |x_1|^p + |x_2|^p + \cdots + |x_n|^p \tag{12}$$

3. Problem statement

Considering that master and slave systems are modeled as (13) and (14) respectively:

$$\begin{cases} D^\alpha y_1(t) = g_1(Y) + \delta G_1(Y,t) + d_1^m(t) \\ D^\alpha y_2(t) = g_2(Y) + \delta G_2(Y,t) + d_2^m(t) \\ \quad\vdots \\ D^\alpha y_n(t) = g_n(Y) + \delta G_n(Y,t) + d_n^m(t) \end{cases} \tag{13}$$

where $\alpha \in (0, 1)$ is the order of fractional derivatives, $Y = [y_1 \ \cdots \ y_n] \in R^n$ is the state vector of the master system, $g_i : R^n \to R$ for $i = 1, \ldots, n$ are given nonlinear functions of Y, $\delta G_i(Y, t)$ and $d_i^m(t)$ for $i = 1, \ldots, n$ are the master system's uncertainty and external disturbances respectively, and

$$\begin{cases} D^\alpha x_1(t) = f_1(X) + \delta F_1(X,t) + d_1^s(t) + u_1(X,Y,t) \\ D^\alpha x_2(t) = f_2(X) + \delta F_2(X,t) + d_2^s(t) + u_2(X,Y,t) \\ \quad\vdots \\ D^\alpha x_n(t) = f_n(X) + \delta F_n(X,t) + d_n^s(t) + u_n(X,Y,t) \end{cases} \tag{14}$$

where $\alpha \in (0, 1)$ is the order of fractional derivatives, $X = [x_1 \ \cdots \ x_n] \in R^n$ is the state vector of slave system, $f_i : R^n \to R$ for $i = 1, \ldots, n$ are given nonlinear functions of $X$and t, $\delta F_i(Y, t)$ and $d_i^s(t)$ for $i = 1, \ldots, n$ are the slave system's uncertainty and external disturbances and finally u_i for $i = 1, \ldots, n$ are control signals.

A large class of fractional chaotic systems can be utilized as master and slave systems in the form of (13) and (14), such as Rossler, Chen, Lu, Liu, Arneodo, Lorenz.

Synchronization error is defined as follows:

$$e_i(t) = x_i(t) - y_i(t), \quad i = 1, \ldots, n \tag{15}$$

Taking α order fractional differentiation from both sides of (15) yields that

$$D^\alpha e_i(t) = D^\alpha x_i(t) - D^\alpha y_i(t), \quad i = 1, \ldots, n \tag{16}$$

Now by replacing (13) and (14) into (16), fractional differential equations that describe the dynamics of the synchronization error are obtained as (17):

$$\begin{cases} D^\alpha e_1(t) = f_1(X) - g_1(Y) + \delta F_1(X,t) - \delta G_1(Y,t) + d_1^s(t) \\ \qquad - d_1^m(t) + u_1(X,Y,t) \\ D^\alpha e_2(t) = f_2(X) - g_2(Y) + \delta F_2(X,t) - \delta G_2(Y,t) + d_2^s(t) \\ \qquad - d_2^m(t) + u_2(X,Y,t) \\ \quad\vdots \\ D^\alpha e_n(t) = f_n(X) - g_n(Y) + \delta F_n(X,t) - \delta G_n(Y,t) + d_n^s(t) \\ \qquad - d_n^m(t) + u_n(X,Y,t) \end{cases} \tag{17}$$

Assumption 1. It is assumed that uncertainties and external disturbances are bounded and differentiable in terms of their arguments and there exist non negative constants F_i^u, G_i^u, D_i^m and D_i^s for $i = 1, \ldots, n$ such that satisfy the following inequalities:

$$\begin{gathered} \left|{}_0^{C}D_t^{1-\alpha}(\delta F_i(X,t))\right| \le F_i^u,\ F_i^u \ge 0,\ i = 1, \ldots, n \\ \left|{}_0^{C}D_t^{1-\alpha}(\delta G_i(X,t))\right| \le G_i^u,\ G_i^u \ge 0,\ i = 1, \ldots, n \\ \left|{}_0^{C}D_t^{1-\alpha}\left(d_i^s(t)\right)\right| \le D_i^s,\ D_i^s \ge 0,\ i = 1, \ldots, n \\ \left|{}_0^{C}D_t^{1-\alpha}\left(d_i^m(t)\right)\right| \le D_i^m,\ D_i^m \ge 0,\ i = 1, \ldots, n \end{gathered} \tag{18}$$