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Stability of localized bulging in inflated membrane tubes under volume control

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Dedicated to Prof. K.R. Rajagopal on the occasion of his 60th birthday.

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ABSTRACT

We study the stability of localized bulging solutions in an inflated hyperelastic membrane tube using an energy stability criterion. We first use this criterion to confirm a previously known result obtained using a dynamic stability criterion, namely that under pressure control all such bulging solutions are unstable. It is then shown that, under volume control, the solutions in the early stages of bulging are unstable whereas those in the later stages of bulging are stable. To be more precise, it is found that the unstable solutions correspond exactly to the snap-back section of the associated pressure–volume diagram.

1. Introduction

When a cylindrical membrane tube is inflated by an internal pressure through air or water pumping, a localized bulge will form when the pressure reaches a critical value p_{cr}. As more air or water is pumped into the tube, the pressure actually drops but the radius at the center of the bulge will increase until it reaches a maximum value r_{max}. With continued inflation, the pressure stays at a constant value p_m, and the bulge spreads in both directions while the radius at the center of the bulge maintains the maximum value r_{max}. This phenomenon is well-known and has been much studied both analytically, experimentally and numerically. For a selection of experimental and numerical studies, we refer to Kyriakides and Chang [18], Pamplona et al. [20], Goncalves et al. [13], Shi and Moita [23], and the references therein. For analytical studies, we mention the bifurcation analysis by Haughton and Ogden [14], and the stability analysis by Shield [24] and Chen [5], all concerned with uniformly inflated states. Two other important analytical studies are those by Chater and Hutchinson [4] who showed how the propagation pressure p_m could be determined by an equal-area rule, and by Yin [25] who gave a detailed characterization of the propagation state (the kinked state).

There have also been a number of studies concerned with the determination of the limiting pressure associated with uniform inflation; see Alexander [1], Benedict et al. [3], and more recently Kanner and Horgan [16]. The existence of such a limiting pressure is usually referred to as limit-point instability, but its connection with the critical pressure p_{cr} has previously not been fully understood analytically although there is experimental evidence showing the equivalence of the two pressures.

This paper is the third of a series of studies devoted to an improved understanding of the above-mentioned inflation process. In the first of this series, Fu et al. [11], the initial bulging/necking was recognized as a bifurcation problem and

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the corresponding bifurcation condition, which can be used to determine the initiation pressure p_{cr}, was derived using two different methods. The bifurcation condition was in fact not new: the expression had previously appeared in the bifurcation analysis of Haughton and Ogden [14] and the stability analysis of Chen [5], but in either case its connection to localized bulging/necking was not fully recognized since the corresponding eigenmode was simply an extra uniform inflation – seemingly unable to describe a localized deformation. It was shown in Fu et al. [11] that the eigenmode is a localized deformation when weakly nonlinear terms were brought in to eliminate the degeneracy in the linear analysis.

In the second of the above-mentioned series, Pearce and Fu [21], our attention was turned to a detailed description of the fully nonlinear bulging/necking solutions and a study of their stability. It turned out that the entire inflation process, described earlier in the first paragraph of this section, could be described by a graphical method aided by some simple analysis. In particular, the exact nature of the transition from radial growth to axial spreading (i.e. from a solitary-wave type solution to a kink-wave type solution) was clarified. A stability analysis was conducted for the case of pressure control using a dynamic stability criterion whereby we apply a small perturbation proportional to $e^{\eta t}$ and investigate whether such a perturbation will grow exponentially. It was found that all the possible bulging/necking solutions were unstable with respect to axially symmetric perturbations.

In the present paper we extend our stability studies to the case when volume is controlled in the inflation process. This is exactly what happened in the experimental study of Kyriakides and Chang [18] where they immersed the membrane tube in water (to eliminate the effect due to gravity) and then inflated the tube by pumping water (which is incompressible) into the tube. They reported that "The initial pressure drop was sudden but, eventually, the deformation process returned to a controlled quasi-static rate of growth". This observation suggests that the initial bulging states immediately after the limiting pressure has been reached is unstable whereas the bulging states at later stages are stable. The main motivation for conducting the present study is to confirm this result. As in the previous two papers in this series, our study is also motivated by our belief that insights derived from the inflation problem will help with our understanding of a variety of other related problems, such as kink-band formation in fibre-reinforced composites (see, e.g., [12]), strain localization (see, e.g., [7]), and stress-induced phase transformations (see, e.g., [9,8]). The latter problems share similar features but analytical results are much harder to come by. We also believe that a full understanding of the inflation process associated with a single-layer membrane tube will shed light on the important problem of continuum-mechanical modeling of aneurysm formation and growth in human arteries, which are multi-layered and have much more complicated constitutive behaviour; see, e.g., Holzapfel et al. [15] and Baek et al. [2].

The energy stability criterion that we employ in the present paper is a criterion that has frequently been used in Continuum Mechanics. For a critical discussion of this method, we refer to the review articles by Ziegler [26] and Knops and Wilkes [17]. For more recent applications, see Chen [5], and Chen and Haughton [6].

The rest of our paper is organized as follows. In the following section, we write down the governing equations and explain briefly how the bulging solutions are determined numerically. In Section 3 we calculate the second variation of the energy functional, the positivity of which ensures local stability of the bulging solutions with respect to the type of perturbations considered. Following the methodology of Chen [5], we minimize the second variation subject to a normalization condition and then solve the resulting linear eigenvalue problem. This is then followed by numerical calculations. The paper is concluded with a summary of our main results and further discussions.

2. Governing equations and the bulging/kink solutions

We consider the problem of inflation of a finite cylindrical membrane tube that is incompressible, isotropic, and hyperelastic with free closed ends. By free ends we mean that the membrane is free to move in both the radial and axial directions at the ends. This differs slightly from the experimental set-up of Kyriakides and Chang [18] where radial expansion at the ends were restricted, but we expect the end effects to be a less important factor than the main features which we aim to capture. The tube is assumed to have uniform thickness H and uniform inner radius R before inflation. We take $R = 1$ in the subsequent analysis; this is equivalent to scaling by R all the variables and constants that have the dimension of length. We assume that the tube always maintains its axi-symmetry so that in terms of cylindrical polar coordinates its deformation can be represented by:

$$r = r(Z), \quad \theta = \Theta, \quad z = z(Z), \quad |Z| < L, \tag{2.1}$$

where Z and z are the axial coordinates of a representative material particle before and after inflation, respectively, L is the half-length before inflation, and r is the radius after inflation. The principal directions of deformation correspond to the lines of latitude, the meridian and the normal to the deformed surface. Hence the principal stretches are given by:

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = \sqrt{r'^2 + z'^2}, \quad \lambda_3 = \frac{h}{H}, \tag{2.2}$$

where a prime represents differentiation with respect to Z, and h denotes the deformed thickness.

We shall derive the equilibrium equations from energy minimization. To this end, we first define an energy functional E through

$$E = \int_{-L}^{L} W(\lambda_1, \lambda_2) 2\pi R dZ - P\left(\int_{-l}^{l} \pi r^2 dz - V_0\right) = \int_{-L}^{L} \left(2\pi R W - \pi P r^2 z' + \frac{P V_0}{2L}\right) dZ, \tag{2.3}$$