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A new accurate numerical method of approximation of chaotic solutions of dynamical model equations with quadratic nonlinearities

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ABSTRACT

In this article the authors describe the method of construction of approximate chaotic solutions of dynamical model equations with quadratic nonlinearities in a general form using a new accurate numerical method. Numerous systems of chaotic dynamical systems of this type are studied in modern literature. The authors find the region of convergence of the method and offer an algorithm of construction and several criteria to check the accuracy of the approximate chaotic solutions.

1. Introduction

Chaotic dynamical systems are difficult to analyze. A close formula giving the solution has not been found yet. Therefore numerical approximations are mandatory in order to follow the motion of a particle driven by a system of a nonlinear differential equation.

Let us consider the system of a differential equations with quadratic nonlinearity

$$\dot{x} = B_0 + B_1 x + \varphi(x), \tag{1}$$

where $x(t) = \left[x_{(1)}(t) \ \ldots \ x_{(m)}(t)\right]^{\mathrm{T}}$ is a m-dimensional real vector function, B_0 is a given m-dimensional real column vector,

$$\varphi(x) = \left[\varphi_{(1)}(x) \ \ldots \ \varphi_{(m)}(x)\right]^{\mathrm{T}},$$

$\varphi_{(p)}(x) = \langle Q_p x, x\rangle$, B_1 and Q_p $(p = \overline{1, m})$ are given real $(m \times m)$ matrices.

Suppose that the system (1) has a bounded solution for $t \geq 0$. Thus, the corresponding trajectory is attracted to the limit trajectory, see [1, pp. 338–340]. Hence, this trajectory determines the behavior of the bounded solutions of the system (1) when time goes to infinity. The limiting trajectory can be a point of equilibrium, a cycle, or can be described by an almost periodic function or can have a more complicated structure such as a strange attractor.

For some dynamical systems (1) the solutions are unstable on their attractors. It causes difficulties applying numerical methods for solving corresponding systems of ordinary differential equa-

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tions (ODE). The problem is not limited to ODE with entire derivatives, but also for dynamical systems governed by fractional derivatives [2–4]. Many researchers use different numerical schemes based on classical methods, e.g. the explicit Euler scheme with the central-difference scheme [5], the Adams scheme [6], the higher derivatives scheme [7], the 4th order Runge–Kutta method [8] and the second-order accurate Adams–Bashforth method [9] for the Lorenz system. However, the above methods cannot be used for (1) due to the instability of chaotic solutions, since the global error increases with the size of the integration interval [10,11]. Strogatz [12, pp. 320-323] computes the estimate of the time T_c when the trajectories of the system (1) decouple critically for the Lorenz system. In [11] the authors present the regression dependence to estimate T_c for the integration step Δt and the order N_o of the numerical scheme

$$T_c(N_o, \Delta t) \approx -2.6 N_o \lg \Delta t \tag{2}$$

for the classical values of parameters of the Lorenz system. They also highlights that the numerical solution converges to different positions of equilibria for various values Δt for the transient chaotic behavior.

Most importantly, the result cannot be improved by decreasing the integration step Δt, since the integration error has an extremum as a function of Δt. This problem can be solved by using high-accuracy calculations [13]. However, this approach restricts the study: on the one hand, the way to decrease the error is narrow (to change Δt and the accuracy of real number representation in order to control the calculation process); on the other hand, the number of operations needed for very small Δt is large. The Runge–Kutta methods can be applied to obtain solutions with a higher accuracy, but the corresponding formulas for $N_o > 6$ are extremely cumbersome [14,15].

In [16] the authors present the multistage spectral relaxation method (MSRM) which differs from the previous direct methods. They use the Chebyshev spectral method to solve the system (1) in the Gauss–Siedel form by an iteration scheme at each subinterval of integration. The advantage is that the accumulation of errors is not as great as it was in the direct methods. Motsa et al. compare the numerical results of MSRM with the piecewise successive linearization method [17]. However, the authors do not study the error of the method as an independent unit and increase the risk of rounding errors.

To find approximate solutions of systems of differential equations, the method of power series (or the method of Taylor series) is sometimes used. In [18–20] this method is used as the Adomian decomposition method (ADM). In those studies, the authors obtain the coefficients of expansion of the solution in a power series for different systems of the form (1) without finding the radius of convergence. The error of the approximate chaotic solution is only compared with the numerical results using the Runge–Kutta methods. Vadasz and Olek [21] also study the dependence of ratio of coefficients of power series with respect to the number of terms in the series.

In this article we consider a modification of the power series method (similar to ADM) for the system (1). An advantage over the general scheme of the Taylor series method is that the expansion coefficients can be rapidly calculated by formulas in comparison to the procedure of symbolic differentiation of the right-hand sides of the system equations (in the nonlinear case huge memory is needed to store the symbol expressions in the calculation of the higher-order derivatives). Also, an estimate of the region of convergence of the power series is obtained, and some criteria for checking the accuracy of the approximate chaotic solutions are given in this article. Recently such an approach has been applied to the Lorenz and Chen systems [22,23]. Here, we generalize our results for the systems in the form (1).

2. Some examples of chaotic systems with quadratic nonlinearities

In this section, we give several examples found in the literature on chaotic systems of the form (1), for which our method can be applied.

2.1. The well known Lorenz system

$$\begin{cases} \dot{x}_{(1)} = \sigma\left(x_{(2)} - x_{(1)}\right), \\ \dot{x}_{(2)} = r x_{(1)} - x_{(2)} - x_{(1)}x_{(3)}, \\ \dot{x}_{(3)} = x_{(1)}x_{(2)} - b x_{(3)}. \end{cases}$$

For this system, the matrices are

$$B_0 = \mathbf{0},\quad B_1 = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix},\quad Q_1 = \mathbf{0},\quad Q_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.2. The Chen system [24,25]

$$\begin{cases} \dot{x}_{(1)} = a\left(x_{(2)} - x_{(1)}\right), \\ \dot{x}_{(2)} = (c-a)x_{(1)} - x_{(1)}x_{(3)} + c x_{(2)}, \\ \dot{x}_{(3)} = x_{(1)}x_{(2)} - b x_{(3)}, \end{cases}$$

for which the matrices are

$$B_0 = \mathbf{0},\quad B_1 = \begin{bmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{bmatrix},\quad Q_1 = \mathbf{0},\quad Q_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.3. The Nose–Hoover oscillator [26]

$$\begin{cases} \dot{x}_{(1)} = x_{(2)}, \\ \dot{x}_{(2)} = -x_{(1)} - x_{(2)}x_{(3)}, \\ \dot{x}_{(3)} = \left(x_{(2)}^2 - 1\right)\Big/ q. \end{cases}$$

In this case, the matrices are

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ -1/q \end{bmatrix},\ B_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ Q_1 = \mathbf{0},\ Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/q & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.4. The Sprott–Jafari system [27]

(we study this example in Section 7 to show the efficiency of our method)

$$\begin{cases} \dot{x}_{(1)} = x_{(2)}, \\ \dot{x}_{(2)} = -x_{(1)} + x_{(2)}x_{(3)}, \\ \dot{x}_{(3)} = x_{(3)} + a x_{(1)}^2 - x_{(2)}^2 - b \end{cases} \tag{3}$$

with corresponding matrices

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix},\quad B_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad Q_1 = \mathbf{0},\quad Q_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$