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Closed-form pricing formula for exchange option with credit risk

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ABSTRACT

In this paper, we study the valuation of Exchange option with credit risk. Since the over-the-counter (OTC) markets have grown rapidly in size, the counterparty default risk is very important and should be considered for the valuation of options. For modeling of credit risk, we use the structural model of Klein [13]. We derive the closed-form pricing formula for the price of the Exchange option with credit risk via the Mellin transform and provide the experiment results to illustrate the important properties of option with numerical graphs.

1. Introduction

Most of the studies of traditional option pricing based on Black-Scholes model [3] have not considered the credit risk of the options. However, there exists the option issuers default risk in the over-the-counter (OTC) market. Specifically, since the financial crisis including the bankruptcy of many financial companies, the default of financial derivatives has become the main concerns of many investors. Since then, the default risk of issuer has been very important in the OTC market. In addition, the size of OTC market has grown rapidly. So, the credit risk should be considered when the options traded in the OTC market are valued.

Johnson and Stulz [10] first proposed the pricing of options with credit risk, which is called *Vulnerable option*. They claimed that the options depend on the liabilities of the option issuers. If the issuers default happens at the maturity, the investor takes all assets of the option issuer. They also considered the correlation between the option issuers asset and the underlying asset. Klein [13] developed the result of Johnson and Stulz by allowing for the proportional recovery of nominal claims in default as well as for the correlation between the default risk of option issuer. In addition, Klein and Inglis [14] dealt with vulnerable options employing the stochastic interest based on Vasicek model. Chang and Hung [7] provided analytic pricing formulae of vulnerable American options under the Black-Scholes framework. However it has been well known that the stochastic volatilities of underlying assets describe well the volatility smile in the real financial market (See, Bonanno et al. [4], [5] and Valenti et al. [18].). In this sense, vulnerable options with stochastic environment have been studied by many researchers in recent years (See, Yang et al. [19], Kim [12] and Lee et al. [16].).

In this paper, we consider the pricing of option with multiple underlying assets and credit risk. Concretely, we study a closed-form pricing formula for Exchange option with credit risk. Exchange option is one of the most popular exotic options in OTC market. Exchange option is a simple option with two underlying assets, which allows its holder to exchange one asset for another. Margrabe [15] first derived pricing formulae for Exchange options with the correlated dynamics of underlying assets. Geman, Karoui, and Rochet [9] used the change of numéraire for closed-form pricing solutions of options. Antonelli, Ramponi, and Scarlatti [1] extended the previous known results under the stochastic volatility. In addition, Cheang and Chiarella [6] dealt with the pricing formulae of Exchange options under the jump-diffusion dynamics. Kim et al. [11] considered Exchange option with the two underlying assets affected by a stochastically changing market environment.

Recently, the valuation of the diverse options has also been studied with the Mellin transform method that is presented by the multiplicative version of the two-side Laplace transform. Panini and Srivastav [17] first used the Mellin transform approach to value the diverse options. Frontczak and Schöbel [8] provided the value of American call options on dividend-paying stocks using the Mellin transform. In addition, Yoon [20] derived a closed-form solution for valuing European options with stochastic interest rate under the Black-Scholes model using the Mellin transform. Yoon and Kim [21] applied the double Mellin transforms to find the pricing formulae of the European vulnerable options under the Hull-White interest rate. We also use the Mellin transforms to obtain the pricing formula of the Exchange option with credit risk.

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The rest of the paper is organized as follows. Section 2 shows that the price of the standard Exchange option is represented by the Mellin transform approach. Section 3 provides the closed-form pricing formula of the Exchange option with credit risk using the double Mellin transform. Section 4 presents the numerical graphs to demonstrate the properties of the prices of option with respect to parameters. Finally, we give the concluding remarks in Section 5.

2. Exchange option : mellin transform approach

In this section, we propose a new approach to price an Exchange option. From the given underlying assets, we formulate the partial differential equation (PDE) for the price of the Exchange option and use the Mellin transform to solve PDE.

We assume that the two underlying assets under the risk-neutral measure Q satisfy the following stochastic differential equation (SDE)

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_t^1,$$
$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_t^2, \tag{1}$$

where r is a constant interest rate, σ_i, $i \in \{1, 2\}$ are the volatilities of asset i and W_t^1 and W_t^2 are the standard Brownian motion with $dW_t^1 dW_t^2 = \rho_{12}dt, \rho_{12} \in [-1, 1]$. Then, for given $S_1(t) = S_1$ and $S_2(t) = S_2$, the no-arbitrage price of an Exchange option C at time t under the measure Q is defined by

$$C(S_1, S_2, t) = E^Q\left[e^{-r(T-t)}C(S_1(T), S_2(T))|S_1(t) = S_1, S_2(t) = S_2\right], \tag{2}$$

with payoff $\max\{S_1(T) - S_2(T), 0\} := (S_1(T) - S_2(T))^+$ as in Margrabe [15].

Let us apply the Feynman-Kac formula to (2). Then we can find that the price $C(S_1, S_2, t)$ satisfies the following system of a PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 C}{\partial S_1^2} + \rho_{12}\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 C}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 C}{\partial S_2^2} + r\left(S_1 \frac{\partial C}{\partial S_1} + S_2 \frac{\partial C}{\partial S_2}\right) - rC = 0, \tag{3}$$

with the terminal condition

$$C(S_1, S_2, T) = (S_1 - S_2)^+$$

and the boundary condition

$$\lim_{S_2 \to \infty} C(S_1, S_2, t) = 0.$$

We now reduce the dimension of PDE for the price of an Exchange option. By the linearity of S_1 and S_2, we have a following property

$$\lambda C(S_1, S_2, t) = C(\lambda S_1, \lambda S_2, t), \tag{4}$$

for any $\lambda > 0$. Taking $\lambda = 1/S_2$ in (4), we have

$$\lambda C(S_1, S_2, t) = S_2 u\left(\frac{S_1}{S_2}, t\right) = S_2 u(S, t), \tag{5}$$

where $S = S_1/S_2$ and $u(S, t) = C(S, 1, t)$. Here, we can find that the derivatives of C in terms of those of u are

$$\frac{\partial C}{\partial S_1} = \frac{\partial u}{\partial S}, \frac{\partial C}{\partial S_2} = u - \frac{S_1}{S_2}\frac{\partial u}{\partial S}. \tag{6}$$

By substituting (6) into (3), we obtain the following PDE for u,

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} = 0, \tag{7}$$

where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$ and the terminal condition is $u(S, T) = (S - 1)^+$.

In order to solve the PDE (7), we adopt the Mellin transform approach. So, we briefly review the definitions and properties of Mellin transform which will be used in this paper. For more details or proofs, see Bertrand et al. [2] and Yoon [20].

Definition 1 (The Mellin transform). For a locally integrable function $f(x), x \in \mathbb{R}^+$, the Mellin transform $\mathcal{M}(f(x), z), z \in \mathbb{C}$ is defined by

$$\mathcal{M}(f(x), z) := \widehat{f}(z) = \int_0^\infty f(x)x^{z-1}dx. \tag{8}$$

If $\widehat{f}(z)$ converges for $a < \Re(z) < b$, where $\Re(z)$ is the real part of z, and c such that $a < c < b$, then the inverse of Mellin transform is given by

$$f(x) = \mathcal{M}^{-1}(\widehat{f}(z)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{f}(z)x^{-z}dz. \tag{9}$$

In fact, the Mellin transform defined by (8) is convergent in $a < \Re(z) < b$ where $f(x) = O(x^{-a})$ as $x \to 0^+$ and $f(x) = O(x^{-b})$ as $x \to \infty$.

Lemma 1 (Convolution of the Mellin transform). *For a locally integrable function f(x) and g(x) on $x \in \mathbb{R}^+$, we assume that the Mellin transforms $\widehat{f}(z)$ and $\widehat{g}(z)$ exist. Then the Mellin convolution of f and g is given by the inverse Mellin transform of $\widehat{f}(z)\widehat{g}(z)$ as follows*

$$f(x) * g(x) = \mathcal{M}^{-1}\left[\widehat{f}(z)\widehat{g}(z); x\right] = \int_0^\infty f\left(\frac{x}{y}\right)g(y)\frac{1}{y}dy, \tag{10}$$

where f(x)∗g(x) is a symbol of the Mellin convolution of f and g.

Lemma 2. *For given complex numbers α and β satisfying $\Re(\alpha) \geq 0$, let $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{f}(z)x^{-z}dz$ with $\widehat{f}(z) = e^{\alpha(z+\beta)^2}$. Then*

$$f(x) = \frac{1}{2}(\pi\alpha)^{-\frac{1}{2}}x^\beta e^{-\frac{1}{4\alpha}(\ln x)^2}$$

holds.

We now provide a pricing formula for an Exchange option by using the properties of the Mellin transform mentioned above.

Theorem 1. *From the relation (5) and the PDE (6) with the terminal condition $u(S, T) = (S - 1)^+$, the closed-form pricing formula of the Exchange option $C(S_1, S_2, t)$ at time t is given by*

$$C(S_1, S_2, t) = S_1\Phi(d_1) - S_2\Phi(d_2). \tag{11}$$

where

$$d_1 = \frac{\ln(S_1/S_2)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}, d_2 = d_1 - \sigma\sqrt{T-t},$$

T is the maturity and $\Phi(a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^a e^{-\frac{1}{2}x^2}dx$.

Proof. Let $\widehat{u}(S, t)$ be the Mellin transform of $u(S, t)$ in the PDE (7). Then, by taking the Mellin transform, the PDE (7) is modified as

$$\frac{\partial \widehat{u}}{\partial t} + \frac{1}{2}(z^2 + z)\widehat{u} = 0. \tag{12}$$

Then, by change of variable $\tau = T - t$, this ordinary differential equation (ODE) leads to the following relation of $\widehat{u}$

$$\widehat{u}(z, \tau) = \widehat{u}(S, T)\exp\left(\frac{1}{2}\sigma^2\tau(z^2 + z)\right) \tag{13}$$

From the inverse Mellin transform, we have

$$u(S, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widehat{u}(S, T)\exp\left(\frac{1}{2}\sigma^2\tau(z^2 + z)\right)S^{-z}dz. \tag{14}$$

If we define

$$B(S, \tau) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\left(\frac{1}{2}\sigma^2\tau(z^2 + z)\right)S^{-z}dz, \tag{15}$$

then, by using the Lemma 2, we have

$$B(S, \tau) = e^{-\frac{1}{8}\sigma^2\tau}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp\left(\frac{1}{2}\sigma^2\tau(z + \frac{1}{2})^2\right)S^{-z}dz$$