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Pre-image entropy for free semigroup actions

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ABSTRACT

The purpose of this study is to indicate fundamental propositions of the pre-image entropy related to a system proposed by Bufetov [Bufetov A. Topological entropy of free semigroup actions and skew-product transformations. J Dyn Control Syst 1999;5:137–143. 1.] for generating free semigroup actions. This study reveals the formula for the pre-image entropy of skew-product transformation with respect to the one-sided shift space. Finally, one example is presented to show how to obtain the pre-image entropy value for the skew-product transformation.

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1. Introduction

Topological entropy can be an indicator of complicated behavior in dynamical systems and ergodic theory. This concept was first defined by Adler et al. [1], and was later given several equivalent definitions by Bowen [5] and Dinaburg [9] for cases in which the domain of the transformation is a metrizable space. The topological entropy of a system can be seen as a quantitative measurement of its orbit complexity in the sense that it presents the rate at which the action of the mapping disperses points. Furthermore, it is understood that topological entropy should be a measure of the uncertainty of the system. For detailed notions and applications on the entropy of dynamical systems, the reader can refer to [17,19,20,22].

Entropy related studies in ergodic theory and dynamical systems became a very active discipline in the early 1970s. There have been many other attempts to find its suitable generalizations for other systems such as groups, graphs, or foliations. For instance, in 1996, with the development of the study of nonautonomous dynamical systems, Koyada et al. [14] introduced and explored the notion of topological entropy for a sequence of continuous self-maps of a compact metric space. Many properties for such dynamical systems were studied in [8,10,11,13,15,21]. Among other studies, Biś and Urbański [3] generalized entropies of a single map to the case of finitely generated semigroup of continuous maps acting on a compact space. Moreover, Biś and Urbański [4] investigated the notion of topological entropy of a semigroup of continuous maps and showed several basic properties. More specifically, Biś and Walczak [2] applied the notion of entropy in a group to hyperbolic groups to study some aspects of dynamics. Prior to that, Friedland [10] used the notion of entropy to study some aspects of the dynamics of graphs and semigroups.

Topological entropy is an important invariant, which presents the complexity of the orbits of map f and is predicted by the forward mapping of f. When a considered mapping f is invertible, we can show that the topological entropy $h_{top}(f^{-1})$ equals $h_{top}(f)$ according to Bowen's consequence. However, if the mapping f is non-invertible, there are several other possibilities which lead to several entropy-like invariants for non-invertible maps. For example, a series of discussions in the development of this type of entropy-like invariants were studied by Hurley [12], Langevin and Walczak [16], Nitecki and Przytycki [18]. In particular, Hurley established the relationships among these entropy-like invariants. Later, Cheng and Newhouse [7] defined two other types of invariants, one "topological" and the other "measure-theoretic" in nature, and provided the variational principle relating these two types of invariants.

The purpose of this paper is to study some properties of pre-image entropy for a free semigroup action defined by Bufetov [6]. The outline of this study is as follows. We first review these notations of topological entropy for the free semigroup action and state the topological entropy formula for the skew-product transformation. Then the definition of pre-image entropy for the free semigroup is provided and investigated. Since the power rule is

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a basic property, this study also demonstrates the inequality of the power rule. In the end of those works, Section 3 establishes the formula among pre-image entropy, one-sided symbolic space and skew-product transformation. Finally, this paper presents the pointwise pre-image entropy value using the example $h_p(f_2, f_3) = \log \frac{5}{2}$, where double map $f_2(z) = z^2$ on the unit circle and triple map $f_3(z) = z^3$ on the unit circle.

2. Free semigroup action

2.1. Preliminary notations

Before studying the pre-image entropy for a free semigroup action, some notations are introduced in this section. The main idea of those notations is presented by that of Bufetov's approximation [6]. Denote by F_m^+ the set of all finite words of symbols $0, 1, \ldots, m-1$. For any $\omega \in F_m^+$, $|\omega|$ stands for the length of ω. If $\omega, \omega' \in F_m^+$, then $\omega\omega'$ will be the word obtained by writing ω' to the right of ω. Obviously, with respect to this law of composition, F_m^+ is a free semigroup with m generators. We write $\omega \le \omega'$ if there exists a word $\omega'' \in F_m^+$ such that $\omega' = \omega''\omega$.

Let $\sum_m = \{0, 1, 2, \ldots, m-1\}^Z$ be the two-sided symbolic space with left shift σ.

A metric on Σ_m is derived from setting

$$d(\omega, \omega') = 1/2^k$$

where $k = \inf\{|n| : \omega_n \neq \omega_n{}'\}$ and $|n|$ is the absolute value of n and $\omega = (\ldots\omega_{-1}\dot{\omega}_0\omega_1\ldots)$, $\omega' = (\ldots\omega_{-1}{}'\dot{\omega}_0{}'\omega_1{}'\ldots) \in \Sigma_m$. This metric space Σ_m is compact. The Bernoulli shift $\sigma : \Sigma_m \to \Sigma_m$ is a homeomorphism of Σ_m given by the formula

$$(\sigma\omega)_i = \omega_{i+1}.$$

Assume that $u \in \Sigma_m, v \in F_m^+$, a and b are integers with $a \le b$. We write $u|_{[a,b]} = v$ if $v = u_a u_{a+1} \cdots u_{b-1} u_b$. Now, we provide the definition of topological entropy of a free semigroup action using separated sets and spanning sets.

Let (X, d) be a compact metric space and assume that $f_0, f_1, \ldots, f_{m-1}$ are continuous maps from X to X. Denote $\mathcal{F} = \{f_0, f_1, \ldots, f_{m-1}\}$. For $\omega = \omega_0\omega_1\ldots\omega_{k-1} \in F_m^+$, set $f_\omega = f_{\omega_0} \circ f_{\omega_1} \circ \cdots \circ f_{\omega_{k-1}}$. A new metric d_ω on X is given by

$$d_\omega(x_1, x_2) = \max_{\omega' \le \omega} d(f_{\omega'}(x_1), f_{\omega'}(x_2)).$$

Let $\varepsilon > 0$, a subset E of X is said to be $(\omega, \varepsilon, \mathcal{F})$-spanning subset if, for $\forall x \in X$, $\exists y \in E$ with $d_\omega(x, y) \le \varepsilon$. The minimal cardinality of a $(\omega, \varepsilon, \mathcal{F})$-spanning subset of X is denoted by $M(\omega, \varepsilon, \mathcal{F})$. Let $\varepsilon > 0$, a subset K of X is called a $(\omega, \varepsilon, \mathcal{F})$-separated subset if, for x_1, $x_2 \in K$, $x_1 \neq x_2$, we have $d_\omega(x_1, x_2) > \varepsilon$. The maximal cardinality of a $(\omega, \varepsilon, \mathcal{F})$-separated subset of X is denoted by $N(\omega, \varepsilon, \mathcal{F})$.

Let

$$M(n, \varepsilon, \mathcal{F}) = \frac{1}{m^n} \sum_{|\omega|=n} M(\omega, \varepsilon, \mathcal{F}), \tag{2.1}$$

and

$$N(n, \varepsilon, \mathcal{F}) = \frac{1}{m^n} \sum_{|\omega|=n} N(\omega, \varepsilon, \mathcal{F}). \tag{2.2}$$

The topological entropy of a free semigroup action is defined by the formula

$$h(\mathcal{F}) = h(f_0, f_1, \ldots, f_{m-1}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon, \mathcal{F}).$$

Equivalently, Bufetov [6] also showed that

$$h(\mathcal{F}) = h(f_0, f_1, \ldots, f_{m-1}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \mathcal{F}).$$

Some stochastic dynamics can be reformulated by a skew-product transformation of two kinds of variables, one of which describes an underlying dynamical system and the other of which describes chaotic dynamics, such as, Bernoulli shifts. Moreover, in the study, Bufetov [6] will be concerned with the relationship between free semigroup actions and skew-product transformations. Let (X, d) be a compact metric space and assume that $f_0, f_1, \ldots, f_{m-1}$ are continuous maps from X to X. Denote $\mathcal{F} = \{f_0, f_1, \ldots, f_{m-1}\}$. Here, the base of skew-product transformations is Σ_m with left shift σ, where its fiber is the compact space X. The map $P : \Sigma_m \times X \longrightarrow \Sigma_m \times X$ is defined by the formula

$$P(\omega, x) = (\sigma\omega, f_{\omega_0}(x)) \quad \text{where } \omega = \cdots\omega_{-1}\dot{\omega}_0\omega_1\cdots.$$

Bufetov [6] formulated the entropy formula for the skew-product transformation as follows,

$$h_{top}(P) = \log m + h(\mathcal{F}),$$

where $h_{top}(P)$ is the topological entropy of the map P.

2.2. Pre-image entropy

The topological entropy predicts the uncertainty of the orbit from the forward mapping. Pre-image entropy takes past behavior into account through the backward mapping. Consider a continuous function f from a compact space X to itself , the concept of pointwise pre-image entropy was introduced in 1990s. Hurley [12] and Nitecki–Przytycki [18] defined pre-image entropy $h_p(f)$ and $h_{\mathbf{m}}(f)$ and Cheng–Newhouse [7] defined $h_{pre}(f \mid \xi^-)$ as follows.

Given a subset $K \subseteq X$, we define the quantity $r(n, \epsilon, K)$ to be the maximal cardinality of (n, ϵ)-separated subsets of K. Following those notations, we have

$$h_p(f) = \sup_{x \in X} \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, f^{-n}(x)),$$

$$h_{\mathbf{m}}(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} r(n, \epsilon, f^{-n}(x)),$$

and

$$h_{pre}(f \mid \xi^-) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X, k \ge n} r(n, \epsilon, f^{-k}(x)).$$

Here, ξ means the point partition and we find $f^{-1}\xi \supseteq f^{-2}(\xi) \supseteq f^{-3}(\xi) \cdots$, then let $\xi^- = \lim_{n\to\infty} f^{-n}(\xi)$. Those pre-image entropies also can be defined by open covers or (n, ϵ)-spanning subsets of X.

Let $\sum_m^+ = \{0, 1, 2, \ldots, m-1\}^N$ be the one-sided symbolic space with left shift σ. From Nitecki–Przytycki [18] and Cheng–Newhouse [7], we can determine that

$$h_p(\sigma) = h_{\mathbf{m}}(\sigma) = h_{pre}(\sigma \mid \xi^-) = \log m.$$

Define the pre-image entropy of the free semigroup generator $\mathcal{F} = \{f_0, f_1, \ldots, f_{m-1}\}$ on the compact space X as follows:

For any $x \in X$ and $\omega = \omega_1\omega_2\ldots\omega_k$ with $|\omega| = k$, define

$$f_\omega^{-1}(x) = f_{\omega_1}^{-1} \circ f_{\omega_2}^{-1} \ldots \circ f_{\omega_k}^{-1}(x)$$

and

$$f_\omega(x) = f_{\omega_k} \ldots f_{\omega_2} f_{\omega_1}(x)$$

Set

$$d_\omega(x_1, x_2) = \max_{\nu \le \omega} d(f_\nu(x_1), f_\nu(x_2))$$

Here, $\nu \le \omega$ means there exists a word α such that $\nu\alpha = \omega$.

Assume $B(\omega, \epsilon, f_0, f_1, \ldots, f_{m-1}, A)$ is the minimum cardinality of the $(\omega, \epsilon, f_0, \ldots f_{m-1})$-spanning subset of ω under the subset $A \subset X$.

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