Chaos, Solitons and Fractals 91 (2016) 299–309

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Robust chaos in a credit cycle model defined by a one-dimensional piecewise smooth map

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ARTICLE INFO

Article history:
Received 5 March 2016
Accepted 19 June 2016

2010 MSC:
37E05
37G35
37N30

Keywords:
One-dimensional piecewise smooth map
Border collision bifurcation
Skew tent map
Homoclinic bifurcation
Robust chaos
Codimension-two bifurcation

ABSTRACT

We consider a family of one-dimensional continuous piecewise smooth maps with monotone increasing and monotone decreasing branches. It is associated with a credit cycle model introduced by Matsuyama, under the assumption of the Cobb-Douglas production function. We offer a detailed analysis of the dynamics of this family. In particular, using the skew tent map as a border collision normal form we obtain the conditions of abrupt transition from an attracting fixed point to an attracting cycle or a chaotic attractor (cyclic chaotic intervals). These conditions allow us to describe the bifurcation structure of the parameter space of the map in a neighborhood of the boundary related to the border collision bifurcation of the fixed point. Particular attention is devoted to codimension-two bifurcation points. Moreover, the described bifurcation structure confirms that the chaotic attractors of the considered map are robust, that is, persistent under parameter perturbations.

1. Introduction

The one-dimensional (1D for short) piecewise smooth (PWS for short) map considered in the present paper defines an important credit cycle model first introduced by Matsuyama in [20]. This model generates *endogenous fluctuations* of borrower net worth and aggregate investment, following the same trend as several micro-founded, dynamic general equilibrium models of financial frictions, in which the steady state is *unstable*, and persistent fluctuations occur *without exogenous shocks* (see, for example, [1,3,21]). Such an approach differs from the basic ideas of a vast majority of the macroeconomics literature on financial frictions that follows the seminal works [6] and [18], and continues to study amplification effects of financial frictions within a setting that ensures the existence of a *stable steady state* toward which the economy would gravitate in the absence of recurring exogenous shocks. In fact, the idea that market mechanisms are *inherently dynamically unstable* can be traced back at least to Goodwin [12]. Recent events have also renewed interest in the hypothesis that financial frictions are responsible not only for amplifying the effects of exogenous shocks but also for causing macroeconomic instability (see, e.g., [17] and [25]).

A detailed description of the Matsuyama model can be found in [20] and [22] (see also [23]). It is defined by a 1D map which consists of upward, downward, and flat branches. Furthermore, as discussed in [23], when the production function is Cobb-Douglas, the map depends on four parameters. The bifurcation structure of the parameter space of this map significantly depends on whether the constant branch is involved into asymptotic dynamics or not. In our companion paper [32] we study in detail the case where all three branches are involved, demonstrating that it is characterized by periodicity regions related to superstable cycles existing due the constant branch, and that these regions are ordered according to the well known *U-sequence* distinctive for unimodal maps (first described in [24], see also [13]), which is adjusted to the considered map.

In the present paper we analyze the dynamics of the map when the constant branch does not participate in the asymptotic dynamics. Such a map belongs to a class of 1D PWS continuous unimodal maps possessing quite complicated dynamics which, depending on the parameters, is characterized by attracting cycles of any pe-

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http://dx.doi.org/10.1016/j.chaos.2016.06.015

riod, as well as cyclic chaotic intervals. The mechanisms governing the transitions between such attractors under parameter variation are already described in our paper [23]. The main purpose of the present work is to give detailed proofs of the related results and to describe the overall bifurcation structure of the parameter space of the map, evidencing the role of codimension-two bifurcation points.

From the point of view of nonlinear dynamics theory the main feature of the considered map is its *non smoothness*. In fact, as we mentioned above, the map is given by two different smooth functions whose definition regions are separated by a *border point* at which the system function is not differentiable. As a result, under variation of a parameter it is possible to observe not only bifurcations typical for 1D smooth maps (such as, for example, flip bifurcation of a fixed point related to its eigenvalue crossing -1, or homoclinic bifurcation related to a contact of a stable and unstable sets of a repelling fixed point), but *border collision bifurcations* (BCB for short) as well, which are characteristic of nonsmooth systems (see [5,14,15,26]). Recall that a BCB occurs when an invariant set, for example, a fixed point or cycle, collides with a border point. The result of such a bifurcation can be a direct transition from an attracting fixed point to a chaotic attractor that is impossible in smooth systems. Such an abrupt transition to chaos in a 1D PWS map can be observed also due to a degenerate bifurcation which is related to the eigenvalue of a fixed point (or cycle) crossing 1 or -1 in presence of a particular degeneracy of the system function. For example, a *degenerate flip bifurcation* (DFB for short) of a fixed point occurs when its eigenvalue crosses -1 and the related branch of the function at the bifurcation value is linear or linear fractional (see [31]). Note that a general bifurcation theory for nonsmooth dynamical systems has not yet such a complete form as the one established for smooth systems. As an important advancement towards such a theory we refer to the books [34], [10]. Examples of PWS models coming from economic applications can be found in [7,9,11,15,28], to cite a few.

As one of the main contributions of the present paper we give the conditions under which abrupt transitions via a BCB from an attracting fixed point to an attracting cycle or to a chaotic attractor are observed. Such conditions are obtained by using a 1D piecewise linear map defined by two linear functions, called *skew tent map*. The dynamics of the skew tent map are completely described depending on the slopes of the linear branches (see [16,19]) that makes it possible to use this map as a *border collision normal form* ([5,27,29,30]).

The skew tent map is used to classify not only the BCB of the fixed point, mentioned above, but BCBs of the attracting n-cycles as well, $n \geq 3$. More precisely, we show that one boundary of the periodicity region related to an attracting n-cycle is associated (at least in a certain neighbourhood) with the so-called *fold BCB*. The crossing of this boundary leads to the appearance of a couple of n-cycles, one attracting and one repelling. This bifurcation is to some extent similar to the smooth fold bifurcation, being, however, not related to an eigenvalue equal to 1. Another boundary of the n-periodicity region is related to the smooth flip bifurcation, sub- or supercritical.

It is known that one more distinctive feature of PWS maps is associated with *robust chaotic attractors* (see [4]), that means that in the parameter space of a PWS map an open region may exist, called chaotic domain, related to chaotic attractors persistent under parameter perturbations. Considering a chaotic attractor which consists of n cyclic intervals, $n \geq 1$, under parameter variation inside a chaotic domain bifurcations can be observed at which the number of intervals constituting the chaotic attractor changes. In particular, a *merging bifurcation* is related to the transition from $2n$- to n-cyclic chaotic attractor. It is caused by the first homoclinic bifurcation of a repelling cycle with negative eigenvalue, located at the immediate basin boundary of the attractor. An *expansion bifurcation* occurs when a chaotic attractor abruptly increases in size filling the complete absorbing interval due to the first homoclinic bifurcation of a repelling cycle with positive eigenvalue (see [2] for details). By using the skew tent map we get the conditions of the homoclinic bifurcations leading to merging and expansion bifurcations in the considered map.

The paper is organized as follows. In Section 2 we describe the map, its fixed points and the conditions of their stability. The parameter region we are interested in is confined by three boundaries. One of them is related to a contact of the absorbing interval with the border point (crossing this boundary the constant branch becomes involved into asymptotic dynamics), and two other boundaries are related to the bifurcations of a fixed point associated with the downward branch of the map. Namely, crossing one of such boundaries a BCB of this fixed point occurs, whose possible results are listed in Section 3 (see Proposition 1) and proved using the skew tent map as a border collision normal form. The second boundary is related to the flip bifurcation described in Section 4 (see Proposition 2). In Section 5 it is discussed the overall bifurcation structure of the parameter space of the considered map, emphasizing the role of codimension-two bifurcation points. Section 6 concludes.

2. Description of the map, its fixed points and their bifurcations

We consider a 4-parameter family of 1D piecewise smooth maps defined as

$$T : w \mapsto T(w) = \begin{cases} T_L(w) = w^{\alpha} & \text{if } 0 < w < w_c, \\ T_M(w) = \left[\dfrac{1}{\mu\beta}\left(1 - \dfrac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} & \text{if } w_c < w < w_\mu, \\ T_R(w) = \beta^{\frac{\alpha}{\alpha-1}} & \text{if } w \geq \max\{w_c, w_\mu\}, \end{cases} \quad (1)$$

where α, β, μ and m are real parameters such that

$$0 < \alpha, \mu < 1, \quad \beta \equiv B\frac{1-\alpha}{\alpha} > 0, \quad 1 < m < \frac{1}{1-\alpha}, \quad (2)$$

w_c and w_μ are the border points defined by

$$w_c^{1-\alpha} = \frac{1}{\mu\beta}\max\left\{1 - \frac{w_c}{m}, \mu\right\}, \qquad w_\mu = m(1-\mu). \quad (3)$$

Map T describes the dynamic trajectory of the entrepreneur *net worth* w in a credit cycle model, first introduced in [20], under the additional assumption that the aggregate production function is Cobb-Douglas (see [23,32]).

In the simplest case map T is defined only by the branches $T_L(w)$ and $T_R(w)$ with the border point $w_c = (w_B)^{1/\alpha}$. The boundary in the parameter space defined by

$$\beta = (m(1-\mu))^{\alpha-1} \quad (4)$$

is related to the appearance of the middle branch in the definition of T. Namely, for $\beta > (m(1-\mu))^{\alpha-1}$ map T can be written in the following form:

$$T : w \mapsto T(w) = \begin{cases} T_L(w) = w^{\alpha} & \text{if } 0 \leq w \leq w_c, \\ T_M(w) = \left[\dfrac{1}{\mu\beta}\left(1 - \dfrac{w}{m}\right)\right]^{\frac{\alpha}{1-\alpha}} & \text{if } w_c < w < w_\mu, \\ T_R(w) = w_B & \text{if } w > w_\mu. \end{cases} \quad (5)$$

Note that T maps $(0, 1]$ into itself, so that we restrict T on $(0, 1]$ from now on.