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# The concept of fractal experiments: New possibilities in quantitative description of quasi-reproducible measurements

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ABSTRACT

In this paper the authors suggest a new conception of the so-called fractal (self-similar) experiment. Under the fractal experiment (FE) one can imply a cycle of measurements that are subjected by the scaling transformations $F(z)\to F(z\xi^m)$ in contrast with conventional scheme $F(z)\to F(z+mT)$ ($m=0,1,\ldots$, M–1), where $z$ defines the controllable (input) variable and can be associated with time, complex frequency, wavelength and etc., $T$ – mean period of time between successive measurements and $m$ defines a number of successive measurements. One can connect a fractal experiment with specific memory effect that arises between successive measurements. The general theory of experiment for quasi-periodic measurements proposed in [1] after some transformations can be applied for the set of the FE, as well. But attentive analysis shown in this paper allows generalizing the previous results for the case when the influence of uncontrollable factors becomes *significant*. The theory developed for this case allows to consider more real cases when the influence of dynamic (unstable) processes taking place during the cycle of measurements corresponding to some FE is becoming essential. These experiments we define as quasi-reproducible (QR) fractal experiments.

The proposed concept opens new possibilities in theory of measurements and numerous applications, especially in different nanotechnologies, when the influence of the scaling factor plays the essential role. This concept allows also to introduce the so-called intermediate model (IM) which can serve as an unified platform for *reconciliation* of the proposed microscopic theory with reliable experiments "refined" from the influence of the random noise and apparatus function. We forced to consider a modified model experiment in order to demonstrate some common peculiarities that can be appeared in real cases. We know only couple of similar examples of experiments that are close to the proposed concept. Mechanical relaxation and dielectric spectroscopy (based on measurements of the complex susceptibility $\varepsilon(j\omega)$) represent the branches of physics related to consideration of mechanical and electric relaxation phenomena in different heterogeneous materials. The dielectric spectroscopy can be considered as an instructive example for better understanding of the proposed concept.

In cases, when the microscopic model is *absent* the results of measurements can be expressed in terms of the fitting parameters associated with the generalized Prony spectrum (GPS) belonging to the IM. The authors do hope that this new approach will find an interesting continuation in various applications of different nanotechnologies.



## 1. Introduction and the formulation of the problem

Here and below $F(z)$ determines a response function that is changed under the control (input) variable $z$. The variable $z$ can coincide with time, complex frequency, pressure, temperature and etc. Under *fractal* measurement we understand the measured response that is registered after scaling of the input variable $z$

$$F(z\xi^m),\ \ m=0,1,\ldots,M-1. \tag{1}$$

*Abbreviations:* (I)FE, (Ideal) fractal experiment; FLSM, the functional least square method; GPS, the generalized Prony spectrum; IM, intermediate model; NCPF, the normalized complex permittivity function; QP, quasi-periodical; QR, quasi-reproducible.

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In contrast to this definition, the conventional measurement is defined as

$$F(x+mT),\ \ m=0,1,...,M-1. \tag{2}$$

If we replace the input variable $x$ for a new and uniform variable $\ln(z)$ ($x\to\ln(z)$) and define the scaling "period" $\ln(\xi)\equiv T$ then one can establish a formal relationship between "fractal" experiment and conventional experiment that includes in itself a set of successive measurements defined by the parameter $M$.

$$F(\ln(z\xi^m))=F(\ln(z)+m\ln(\xi))\equiv F(x+mT),\\ m=0,1,...,M-1. \tag{3}$$

This formal replacement in spite of its simplicity has a deep physical meaning. The dielectric spectroscopy was the first spectroscopy where the fractal properties of the matter were registered in the form of empirical power-law relationships. At the first time the power-law relationship was suggested empirically in the pioneer paper [2] of the Cole's brothers in year 1941 for description of dielectric relaxation in polar liquids. For the fitting of the dielectric loss spectrum in frequency region (which has a symmetrical broadened peak) the Cole's suggested to introduce the convenient stretching power-law exponent $\alpha$ ($0<\alpha\le 1$) into the conventional relaxation Debye's law

$$\hat{\varepsilon}_{CC}(z)=\hat{\varepsilon}_D(z^\alpha)=\frac{1}{1+(z/\omega_p)^\alpha}, \tag{4}$$

where $z=i\omega$,

$$\hat{\varepsilon}(z)=\frac{\varepsilon(z)-\varepsilon_\infty}{\varepsilon(0)-\varepsilon_\infty}. \tag{5}$$

The function $\hat{\varepsilon}(z)$ defines the normalized complex dielectric permittivity (NCDP), $\varepsilon(z)$–the complex dielectric permittivity (CDP), $\varepsilon_\infty$–the high-frequency limit of the CDP, $\omega_p$ is the characteristic loss peak frequency. For description of the asymmetric and broadened loss peaks two researches Havriliak and Negami (HN) [3] introduced an additional stretching exponent $\beta$ ($0<\beta\le 1$) and obtained another popular empirical expression

$$\hat{\varepsilon}_{HN}(z)=\frac{1}{\left(1+(z/\omega_p)^\alpha\right)^\beta}, \tag{6}$$

which is known in dielectric spectroscopy as the HN formula. In the partial case $\alpha=1$ expression (2) is transformed to the well-known Cole-Davidson (CD) expression for the NCDP [4]. We want to stress here that usual presentation of expressions (4) and (5) as a function of the frequency $\omega$ is *wrong*. In order to see anomalous behavior of the NCDP functions all experimentalists use the uniform decimal $\log(\omega)$ scale for presentation the measured data. Only this logarithmic presentation allows "stretching" the frequency scale and to notice the power-law behavior of expressions (4) and (5) especially in low-frequency scale. But we want to mark here that all dielectric measurements are repeated in the conventional manner (2). In order to come back to fractal measurements we should present expression (6) in the following form:

$$\hat{\varepsilon}_{HN}(\ln(z)+m\ln(\xi))=\frac{1}{\left(1+\exp\left(\alpha\left[\ln\left(\frac{z}{\omega_p}\right)+m\ln\xi\right]\right)\right)^\beta} \tag{7}$$

Here we omit the further modifications of expressions (4) and (6) related to the description of the NCDP. The corresponding expressions one can find in the recent papers related to dielectric spectroscopy [5–11]. For us it is important to stress only the following fact. Presentation some data in logarithmic scale allows to fix more interesting details in comparison with the conventional presentation when some important details in behavior of different functions are remained *unnoticeable*. As it follows from expression (3) any fractal measurement requires the presentation in the logarithmic scale as the *obligatory* one. So, probably, the idea of presentation of the NCDP function in the uniform logarithmic scale could be used for *all* fractal experiments. In order to be more objective we should remind also the mechanical relaxation phenomenon [12] where the logarithmic presentation of the measured frequencies is used as the conventional procedure.

The basic problem that is considered in this paper can be formulated as: *how to modify the conventional scheme of measurements and adjust it for consideration of the properties of the quasi-reproducible (QR) measurements?* Under QR-measurements we understand the measurements when the influence of uncontrollable factors is essential, while in quasi-periodic measurements [1] these random factors are remained relatively stable. The formulation of the problem in this manner allows to find a "universal" fitting function corresponding to the intermediate model (IM). This IM can serve as a unified platform (or "bridge") for *reconciliation* of the proposed microscopic theory with the "refined" experiment, when the influence of the external uncontrollable factors and the apparatus function is inspected. We are going to consider these two cases because of their importance for many practical applications.

We should remind here that the fractal (self-similar) concept plays an essential role in modern natural sciences. At present time, the following tendency is observed clearly in modern sciences: (a) the numerous proofs discovered by many researches that the self-similarity concept is really general and confirmed in many natural sciences and (b) the self-similarity concept enriched the "mathematical physics" of fractals in the form of the intensive development of the fractional calculus and its applications. Before these two trends (including many papers and books that cannot be listed here) that are developed almost independently from each other, many researches tried to establish new relationships between fractals and fractional calculus. Here we want to mark especially the book [12] and papers [13–15], where new links between these two trends have been established. This paper put forward another concept: the *imposed* fractal experiments that can create the desired self-similarity during the process of the realized measurements. The authors hope that these FEs will help to understand deeper the general concept of *fractality* that can appear in rather unexpected places and these type of experiments will be claimed in different nanotechnologies.

## 2. The general concept of the fractal experiment

In this section we consider the general concept of fractal experiments for two cases: (a) quasi-periodical and (b) quasi-reproducible measurements. The case (a) uses partly the results obtained earlier in paper [1] while the consideration of the second case (b) is original.

### 2.1. The case of quasi-periodic measurements

Let us consider the definition of *ideal* fractal experiment (IFE). Under this experiment we understand the case when any scaling relatively the fixed value $\xi$ remains the result of the measurement *invariant*

$$F\left(z\xi^{m+1}\right)=F(z\xi^m),\ \ m=0,1,...,M-1. \tag{8}$$

For further purposes it is convenient to present this condition in the form

$$F(\ln z\pm(m+1)\cdot\ln\xi)=F(\ln z\pm m\cdot\ln\xi). \tag{9}$$

The solution of this functional equation is known and can be presented in the form of the segment of the log-periodic Fourier