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Turing instability for a competitor-competitor-mutualist model with nonlinear cross-diffusion effects

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ABSTRACT

This paper deals with a strongly coupled reaction-diffusion system modeling a competitor-competitor-mutualist three-species model with diffusion, self-diffusion and nonlinear cross-diffusion and subject to Neumann boundary conditions. First, we establish the persistence of a corresponding reaction-diffusion system without self- and cross-diffusion. Second, the global asymptotic stability of the unique positive equilibrium for weakly coupled PDE system is established by using a comparison method. Moreover, under certain conditions about the intra- and inter-species effects, we prove that the uniform positive steady state is linearly unstable for the cross-diffusion system when one of the cross-diffusions is large enough. The results indicate that Turing instability can be driven solely from strong diffusion effect of the first species (or the second species or the third species) due to the pressure of the second species (or the first species).

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In this paper, we are interested in a strongly coupled reaction-diffusion system

$$\begin{cases} u_{1t}-\Delta[(d_1+\alpha_{11}u_1+\alpha_{12}u_2+\frac{\alpha_{13}}{\beta+u_3})u_1]=u_1(a-u_1-\frac{\delta u_2}{1+mu_3}) \quad \text{in} \quad \Omega\times(0,\infty), \\ u_{2t}-\Delta[(d_2+\alpha_{21}u_1+\alpha_{22}u_2)u_2]=u_2(b-u_2-\eta u_1) \quad \text{in} \quad \Omega\times(0,\infty), \\ u_{3t}-\Delta[(d_3+\frac{\alpha_{31}}{\gamma+u_1}+\alpha_{33}u_3)u_3]=u_3(c-\frac{u_3}{L_0+nu_1}) \quad \text{in} \quad \Omega\times(0,\infty) \end{cases} \tag{1.1}$$

with initial and boundary value conditions

$$\frac{\partial u_1}{\partial\nu}=\frac{\partial u_2}{\partial\nu}=\frac{\partial u_3}{\partial\nu}=0 \quad \text{on} \quad \partial\Omega\times(0,\infty),$$
$$u_i(x,0)=u_{i0}(x)\geq(\not\equiv)0 \quad \text{in} \quad \Omega \text{ for } i=1,2,3, \tag{1.2}$$

where ν is the unit outward normal to $\partial\Omega$, $d_i(i=1,2,3)$, a, b, c, δ, m, η, L_0, n, β and γ are all positive constants, $\alpha_{ii}(i=1,2,3)$, α_{12}, α_{13}, α_{21} and α_{31} are nonnegative constants. a, b and c are intrinsic growth rates of the three species, respectively, while δ, m, η, L_0 and n describe inter-species interactions. This system represents a model which involves interacting and migrating in the same habitat Ω among a competitor u_2, a competitor-mutualist u_1 and a mutualist u_3. The populations are not homogeneously distributed due to the consideration of diffusions and cross-diffusions. For more biological meaning of the parameters, one can make a reference to [15,17].

The spatially homogeneous ODEs of (1.1) was initiated by Rai et al. [15]. Sufficient criteria for the boundedness of global solution and the local stability or instability of various equilibria were established. Zheng [27] then extended and considered the corresponding reaction-diffusion system ($\alpha_{11}=\alpha_{22}=\alpha_{33}=\alpha_{12}=\alpha_{13}=\alpha_{21}=\alpha_{31}=0$ in (1.1)) under Dirichlet and Neumann boundary conditions. He discussed the local stability of positive equilibrium and the stability of various semitrivial steady states. Xu [23] investigated some sufficient conditions under which there is no non-constant positive steady state for the same weakly coupled reaction-diffusion model. In addition, the asymptotic behavior of positive solutions for periodic system was studied by A. Tineo [20] and Fu et al. [4]. Y. Du [3] also discussed the existence of positive periodic solutions of the corresponding Dirichlet problem by using degree and bifurcation theories.

As for the strongly coupled system of this competitor-competitor-mutualist model, there are also some important results. When $\alpha_{12}=\alpha_{13}=\alpha_{31}=0$, Chen et al. [2] obtained some existence and non-existence results concerning non-constant positive steady-states for the Neumann problem by using Leray–Schauder degree theory. Recently, under Dirichlet boundary value conditions, Li et al. [9] discussed the existence of positive solutions to a competitor-competitor-mutualist model with another type of strongly coupled terms by Schauder fixed point theory. Their

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results show that the system possesses at least one coexistence state if cross-diffusions and cross-reactions are weak.

The weakly coupled system of (1.1) don't consider either the fact that competitors and mutualist naturally develop strategies for survival or the fact that the distribution of population is usually not homogeneously. To take into account the intra-specific and inter-specific population pressures between two competitors and mutualist, we introduce self- and cross-diffusions. The dispersal terms can be written as

$$\mathrm{div}\left\{(d_1+2\alpha_{11}u_1+\alpha_{12}u_2+\frac{\alpha_{13}}{\beta+u_3})\nabla u_1+\alpha_{12}u_1\nabla u_2+\frac{-\alpha_{13}u_1}{(\beta+u_3)^2}\nabla u_3\right\},$$
$$\mathrm{div}\{\alpha_{21}u_2\nabla u_1+(d_2+\alpha_{21}u_1+2\alpha_{22}u_2)\nabla u_2\},$$
$$\mathrm{div}\left\{\frac{-\alpha_{31}u_3}{(\gamma+u_1)^2}\nabla u_1+(d_3+\frac{\alpha_{31}}{\gamma+u_1}+2\alpha_{33}u_3)\nabla u_3\right\}.$$

The terms

$$d_1+2\alpha_{11}u_1+\alpha_{12}u_2+\frac{\alpha_{13}}{\beta+u_3},\quad d_2+\alpha_{21}u_1+2\alpha_{22}u_2,\quad d_3+\frac{\alpha_{31}}{\gamma+u_1}+2\alpha_{33}u_3$$

represent self-diffusions and the terms

$$\alpha_{12}u_1,\quad \frac{-\alpha_{13}u_1}{(\beta+u_3)^2},\quad \alpha_{21}u_2,\quad \frac{-\alpha_{31}u_3}{(\gamma+u_1)^2}$$

represent cross-diffusions. Here $\alpha_{12}u_1>0$ (or $\alpha_{21}u_2>0$) implies that the flux of u_1 (or u_2) is directed toward the decreasing population density of u_2 (or u_1), so the two competitors avoid each other. While $\frac{-\alpha_{13}u_1}{(\beta+u_3)^2}<0$ (or $\frac{-\alpha_{31}u_3}{(\gamma+u_1)^2}<0$) implies that the flux of u_1 (or u_3) is directed toward the increasing population density of u_3 (or u_1), i.e., the two mutualists chase each other.

After add these items, model (1.1) means that, in addition to the dispersive force, the diffusion also depends on population pressure from other species. Thus, the populations in (1.1) are not homogeneously distributed due to the consideration of self- and cross-diffusions.

The roles of diffusion and cross-diffusion in the modeling of biological processes have been extensively studied in literature. Starting with Turing's seminal work [21], diffusion and cross diffusion have been observed as causes of the spontaneous emergence of ordered structures, called patterns, in a variety of nonequilibrium situations. Diffusion-driven instability, also called Turing instability, has also been verified empirically in some chemical and biological models [1,5,19,22]. For some systems with cross-diffusion, we can learn that cross-diffusion may be helpful to create linear instability as well as non-constant positive steady-state solutions for corresponding ecosystems, for example [10–13,16,18]. Recently, Guin [7] investigated a mathematical model of predator-prey interaction subject to self and cross-diffusion and found that the effects of self-diffusion as well as cross-diffusion play important roles in the stationary pattern formation of the model which concerns the influence of intra-species competition among. Hoang et al. [8] considered a general n-species reaction-diffusion system. Under some assumptions of diffusion and reaction matrices, linear instability and dynamical instability for the uniform steady state were discussed by linearization and a bootstrap lemma. These results show that the cross-diffusion systems are capable of producing much more complex dynamics than the corresponding diffusion system, which can provide theoretical basis for numerical simulation of various spatial patterns, such as spotted, spots-stripes mixtures, stripe-like, oscillatory patterns, and so on.

In recent years, researches on the existence of non-constant steady states and patterns formation for strongly coupled reaction-diffusion systems arising from population dynamics have been mainly focused on the models with linear cross-diffusion [2,5,12,13,19,22], and relatively little research has been conducted to the mutli-species models with nonlinear cross-diffusion terms (for example, [6] and [9] for species coexistence). In our study, a reaction-diffusion system of competitor-competitor-mutualist model with diffusion, self-diffusion and nonlinear cross-diffusion is considered. Our objective is to discuss the roles of diffusion, self-diffusion and cross-diffusion in stationary patterns formation for model (1.1), (1.2). We prove that cross-diffusion α_{12}, α_{21} or α_{31} can destabilize a uniform positive equilibrium which is stable for the ODE system and for the weakly coupled reaction-diffusion system. As a result, we find that under certain conditions, the effect of cross-diffusion can arouse stationary patterns while diffusion and self-diffusion fail to do so. Our results exhibit some interesting combining effects of cross-diffusion, competition, mutualism and intra-species interactions on the stability and instability of positive equilibrium.

The paper is organized as follows. In Section 2, the persistent property for reaction-diffusion system with no self- and cross-diffusion is discussed by using a comparison method. Under the same condition on locally stability in [27], we obtain the globally asymptotic stability of the uniform positive steady state for weakly coupled reaction diffusion system. In Section 3, we investigate the linear stability of uniform positive steady state for ODEs and reaction-diffusion system with no or with one cross-diffusion and the effect of cross-diffusion α_{12}, α_{21} or α_{31} on the appearance of Turing instability.

2. Persistence and global asymptotic stability for the PDEs without self- and cross-diffusion

By solving the equations

$$a-u_1^*-\frac{\delta u_2^*}{1+mu_3^*}=0,\quad b-u_2^*-\eta u_1^*=0,\quad c-\frac{u_3^*}{L_0+nu_1^*}=0,$$

it is easy to know that problem (1.1) has a unique positive equilibrium

$$\mathbf{u}^*=(u_1^*,u_2^*,u_3^*)^{\mathrm{T}}=(u_1^*,b-\eta u_1^*,c(L_0+nu_1^*))^{\mathrm{T}}.$$

if

$$a(1+mcL_0)>\delta b,\quad b>\eta u_1^*, \tag{2.1}$$

where $u_1^*=\frac{-(1+mcL_0-amnc-\delta\eta)+\sqrt{(1+mcL_0-amnc-\delta\eta)^2+4(a+amcL_0-\delta b)}}{2mnc}$.

In this section, we always assume that $\alpha_{ij}=0$. We will show that any nonnegative classical solution $\mathbf{u}(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))^{\mathrm{T}}$, $u_i\in C^{2,1}(\Omega\times(0,T))\cap C(\overline{\Omega}\times(0,T))(0<T<+\infty)$ of (1.1) without self- and cross-diffusion lies in a certain bounded region, and even converges to the positive equilibrium $\mathbf{u}^*=(u_1^*,u_2^*,u_3^*)^{\mathrm{T}}$ as $t\to+\infty$ for all $x\in\Omega$.

Theorem 2.1. *Suppose inequalities*

$$a(1+mcL_0)>\delta b,\, b>\eta a \tag{2.2}$$

are fulfilled. Then system (1.1) is persistent.

Proof. The proof is similar to the arguments in Theorem 2.2 of [24]. From the maximum principle for parabolic type equation, all solutions of (1.1) are nonnegative since the initial value is nonnegative. From the first equation in (1.1), we can obtain

$$u_{1t}-d_1\Delta u_1=u_1(a-u_1-\frac{\delta u_2}{1+mu_3})\le au_1(1-\frac{u_1}{a}).$$

Then $u_1\le v$ from the comparison principle for parabolic equation, here v is the solution of

$$v_t-d_1\Delta v=av(1-\frac{v}{a}),\quad \frac{\partial v}{\partial \nu}=0,\quad v(x,0)=u_{10}(x)\ge(\not\equiv)0.$$