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Some properties of models for generalized Oldroyd-B fluids

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ABSTRACT

A class of generalized Oldroyd-B fluids is studied with reference to properties that are relevant in obtaining energy estimates. The generalization takes the form of a viscosity that depends on shear rate, with shear-thinning behaviour being of particular interest. Conformation stress tensors for the generalized problem are defined, and the conditions for positive-definiteness of the stress, and for its determinant to exceed unity, are obtained. It is shown that these properties, established for standard Oldroyd-B fluids, carry over provided that an expression depending on the Weissenberg number, a relaxation time, and the rate of change of viscosity, is positive. Various alternative energy estimates are obtained. The consequences of the positivity constraint are investigated for a specific viscosity function due to Yeleswarapu et al. [K.K. Yeleswarapu, M.V. Kameneva, K.R. Rajagopal, J.F. Antaki, The flow of blood in tubes: theory and experiment, Mechanics Research Communications 25(3) (1998) 257–262], first in general and then for a set of special flows.

1. Introduction

There is a growing literature on the mathematical and numerical aspects of viscoelastic fluid flows, as is evident from the works [13,19,20] and the references therein. Of particular interest, in addition to considerations of well-posedness (see, for example [9,10]), are the construction and analysis of suitable functionals for the energy of the system. These functionals provide information about the stability of viscoelastic flows, as well as a framework for the implementation of numerical schemes which inherit desirable properties such as stability. Relevant works in this regard include [7,15,16].

The focus of the works referred to is generally on the systems of equations for Oldroyd-B and related fluids governed by a constitutive law of differential type for the extra stress. An important generalization of these models is to the case in which the viscosity is not constant, but rather depends on the magnitude of the rate of deformation. Such a generalization is of great practical importance as it provides the basis for models of a range of natural and synthetic fluids: examples are to be found among biological as well as polymeric fluids.

A key physical feature that needs to be captured in these cases is shear thinning or -thickening. The simplest form in which a variable viscosity may be accounted for is by replacing the constant viscosity in the equations for Newtonian fluids by one that is a function of the flow. Such models are known as generalized Newtonian fluids. The generalized Oldroyd-B fluids represent another extension: these are particularly relevant in the modelling of biological fluids such as blood, which exhibit both shear-thinning as well as viscoelastic behaviour. A comprehensive review of models for blood may be found in [22].

A generalized Oldroyd-B model applicable to blood flow has been developed in [21]. The model incorporates shear-thinning, and has an empirical basis. In later work [1] a model that incorporates viscoelastic effects and shear-thinning

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and which has a thermodynamic basis has been developed. This falls within a broad class of models with shear-dependent viscosity studied in [2], in which the existence and uniqueness of a solution are established for small and suitably regular data, and subject to mild conditions on the regularity of the viscosity function and its derivatives.

Considerations of energy stability of viscoelastic fluid models have also received growing attention. For the particular case of Oldroyd-B fluids, free energy estimates have been derived for various definitions of the energy associated with the extra stress. It is common to make use of the conformation tensor. This is done in [16], for example, where the energy term for the stress involves the trace of the conformation tensor, its positive-definiteness being derived on the basis of its interpretation as an ensemble average of second moments of configuration vectors for suspensions of Hookean dumbbells. This is also the approach adopted in [11].

In [7] the primary objective is to develop fully discrete numerical schemes based on the use of finite elements in space. For this purpose a free energy is defined, shown to be dissipative for the continuous problem, and numerical schemes are constructed which are shown to inherit this property. The energy term for the conformation stress $\boldsymbol{\sigma}$ takes the form $\mathrm{tr}(\boldsymbol{\sigma} - \ln \boldsymbol{\sigma} - \boldsymbol{I})$, which is shown to be positive-semidefinite provided that $\boldsymbol{\sigma}$ is symmetric positive-definite. This condition in turn is shown to hold provided that $\boldsymbol{\sigma}$ is positive-definite at time $t = 0$.

It is clear then that properties of positive definiteness of the conformation tensor or expressions involving it are important considerations in studies of the stability of Oldroyd-B and related models. The question arises as to the feasibility of extending these approaches to the case of generalized Oldroyd-B fluids. Specifically, the task is to define an appropriate energy and to determine the conditions under which it exhibits dissipative properties for situations in which the viscosity depends on the flow. The purpose of this contribution is to address this question.

Models of the form studied in [2] are considered. First, a conformation stress appropriate to the problem is defined, and the conditions under which it is positive-definite, or has a determinant greater than unity, are explored. A quantity ζ involving the Weissenberg number and the material rate of change of the viscosity function is defined, and it is shown that the properties established for classical Oldroyd-B fluids carry over provided that $\zeta > 0$. For one particular definition of energy there is also a constraint on the magnitude of ζ.

In order to make concrete the constraints that follow from the condition on ζ the viscosity function proposed in [21] is selected as a model function and the consequences of the constraint explored in terms of this function, first generally and then for some simple flows. While the Yeleswarapu function does not possess a thermodynamic basis it does capture the shear-thinning behaviour of blood over a large range of shear rates. Furthermore, it turns out to endow the function ζ with interesting properties in relation to the bounds on Weissenberg number that would ensure stability.

The plan of the rest of this work is as follows. In Section 2 the governing equations for generalized viscoelastic fluids are presented and then rendered in dimensionless form. The conformation tensors are introduced in Section 3, their relevant properties discussed, and the problem is transformed to one involving the conformation tensor. Section 4 is devoted to an exploration of properties such as positive-definiteness and the conditions under which it holds. The implications for energy estimates are addressed in Section 5, and in Section 6 the consequences of the constraint on the quantity ζ are presented and discussed. In Section 7 some concluding remarks are made.

2. The mathematical model

We consider the classical Oldroyd-B fluid with viscosity dependent on the rate of deformation. The fluid, confined to an open bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with a smooth boundary $\partial\Omega$, is governed by the equations

$$\begin{aligned} &\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \operatorname{div} \boldsymbol{S} + \boldsymbol{f}, \\ &\operatorname{div} \boldsymbol{u} = 0, \\ &\boldsymbol{S} + \lambda_1 \overset{\triangledown}{\boldsymbol{S}} = \eta(\dot{\gamma})\boldsymbol{A}_1 + \eta_r \lambda_2 \overset{\triangledown}{\boldsymbol{A}}_1 . \end{aligned} \tag{2.1}$$

Here $\boldsymbol{u}$, p, $\boldsymbol{S}$ denote, respectively, the velocity, the hydrostatic pressure, and the (symmetric) extra-stress tensor. The body force is $\boldsymbol{f}$ and the stretching tensor $\boldsymbol{A}_1$ is given by $\boldsymbol{A}_1 = \boldsymbol{L} + \boldsymbol{L}^T$, with $\boldsymbol{L} = \nabla \boldsymbol{u}$ being the velocity gradient. Thus $\boldsymbol{A}_1$ is twice the rate of deformation $\boldsymbol{D}$. The quantity $\dot{\gamma}$ is a measure of the magnitude of the rate of deformation, and is defined by

$$\dot{\gamma} = \sqrt{\frac{1}{2}\boldsymbol{A}_1 : \boldsymbol{A}_1}. \tag{2.2}$$

Here and henceforth the scalar product of two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A}:\boldsymbol{B}$, or $A_{ij}B_{ij}$ in index form, the summation convention being implied for repeated indices. This scalar quantity is often referred to as the shear rate in shearing flows, or the extensional rate in extensional flows. For convenience the term 'shear rate' will be used henceforth when referring to $\dot{\gamma}$.

The function $\eta > 0$ is the flow-dependent viscosity function and η_r its asymptotic value as $\dot{\gamma} \to r = 0$ or ∞ corresponds to shear-thinning or -thickening, respectively. The fluid density $\rho > 0$ is dimensionless constant and $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ are constants of the relaxation and retardation times, respectively, with dimensions of times.

The linear operators D/Dt and $(\overset{\triangledown}{\cdot})$ denote the material and the upper convected derivatives, respectively, and are defined by