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Relations of the almost average shadowing property with ergodicity and proximality

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ABSTRACT

The aim of this paper is to relate the almost average shadowing property (ALASP) with various variants of ergodicity and with some other dynamical properties. We also study the relation of the ALASP with proximality.

1. Introduction

Throughout this paper a dynamical system is a pair (X, f), where X is a compact metric space with metric d and $f: X \to X$ is a continuous map.

Pseudo-orbits are important tools for investigating properties of discrete dynamical systems, for instance, they can detect mixing and recurrent behaviours of the system which may not be evident by studying actual orbits. One useful application of pseudo-orbits is obtained in neuroscience [11]. In mathematics, a well known application of pseudo-orbits is shadowing. The theory of shadowing has importance in both qualitative theory and in the theory of numerical methods. It also plays a significant role in ergodic theory [8].

Pseudo-orbits are obtained during the numeric simulations of an orbit of a dynamical system, however, a system having the shadowing property forces these simulated orbits to follow true orbits of the system. Since the introduction of the classical shadowing property, several variations of this concept have been defined and extensively studied in the literature [5,14,16,18]. However, all these different notions of shadowing share a common motivation of studying the closeness of approximate and exact orbits of a dynamical system by modifying the notion of pseudo-orbit along with the definition of tracing points. In [4], Blank introduced the concept of average pseudo-orbit and later the notion of average shadowing property and its variants have been further studied, purely in topological sense, by several authors [10,12,13,15]. The motivation of their definitions come from the situations where instead of an exact error bound in each step of a simulated orbit one gets a small average error in long runs of the orbit. In [7], authors have introduced the notion of almost average shadowing property (ALASP) and have studied its recurrence and iterative properties.

Another two fundamental concepts in the theory of topological dynamics are proximality and distality. There are several significant theorems related to these concepts including Furstenberg's structure theorem of distal systems [9].

The main focus of this paper is to relate the ALASP with some other interesting dynamical properties. Our main findings are as follows. In Section 3, we prove that in a dynamical system, a surjective map having the ALASP and the shadowing property has dense set of minimal points. It is also shown that such a map is totally strongly ergodic and hence totally topologically ergodic. We also give for a map having the shadowing property some equivalent conditions to have the ALASP. Here we also give some examples and counterexamples in support of some of our results. In Section 4, we obtain that in an infinite dynamical system, a map having the ALASP has plenty of proximal pairs other than the trivial ones which in turn gives that notions of the ALASP and distality

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can not stay together. We also obtain that in a dynamical system having two distinct minimal sets, a map having the ALASP can not be almost distal.

2. Preliminaries

We write $\mathbb{R}$ for the set of real numbers, $\mathbb{Q}$ for the set of rational numbers, $\mathbb{N}$ for the set of positive integers and $\mathbb{Z}_+$ for the set of nonnegative integers. Let (X, f) be a dynamical system. The *f-orbit* of a point $x \in X$, denoted by $O_f(x)$, is given by the set $O_f(x) = \{f^n(x) : n \geqslant 0\}$. The *ω-limit set* of a point $x \in X$, denoted by $\omega_f(x)$, is the set of limit points of $O_f(x)$. For $\delta > 0$, a sequence $\{x_i\}_{i \geq 0}$ in X is called a δ-pseudo-orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for every $i \geq 0$. The map f is said to have the *shadowing property* if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-pseudo-orbit $\{x_i\}_{i \geq 0}$ of f is ϵ-shadowed by some point $z \in X$, that is, $d(f^i(z), x_i) < \epsilon$ for every $i \geq 0$. It is well known that if f has the shadowing property, then $f \times f$ and f^k have the shadowing property for every $k > 1$ [1].

The map f is said to have the *asymptotic average shadowing property* (AASP) [10] if every asymptotic average pseudo-orbit $\{x_i\}_{i \geq 0}$ of f, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) = 0$$

is asymptotically shadowed in average by some point $z \in X$, that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) = 0.$$

For $\delta > 0$, a sequence $\{x_i\}_{i \geq 0}$ in X is called an almost δ-average-pseudo-orbit of f if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f(x_i), x_{i+1}) < \delta.$$

The map f is said to have the *almost average shadowing property* (ALASP) [7] if for every $\epsilon > 0$, there is a $\delta > 0$ such that every almost δ-average-pseudo-orbit $\{x_i\}_{i \geq 0}$ of f is ϵ-shadowed in average by some point $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

For $\delta > 0$, a sequence $\{x_i\}_{i \geq 0}$ in X is called a δ-average-pseudo-orbit of f if there is an integer $N = N(\delta) > 0$ such that for all $n \geq N$ and all $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$

The map f is said to have the *average-shadowing property* (ASP) [4] if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i \geq 0}$ of f is ϵ-shadowed in average by some point $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

Recall that for $\delta > 0$ and $x, y \in X$, a *δ-chain* of f from x to y of *length* $n \in \mathbb{N}$ is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ satisfying $d(f(x_i), x_{i+1}) < \delta$ for $0 \leqslant i \leqslant n-1$. The map f is said to be *chain transitive* if for any $\delta > 0$ and any pair $x, y \in X$, there is a δ-chain of f from x to y and it is said to be *totally chain transitive* if each iterate f^k, $k \in \mathbb{N}$, is chain transitive. Also, the map f is said to be *chain mixing* if for any $\delta > 0$ and any pair $x, y \in X$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there is a δ-chain of f from x to y of length n [17].

For nonempty open subsets U, V of X and $x \in X$, we write $N(U, V) = \{n \in \mathbb{Z}_+ : f^n(U) \cap V \neq \emptyset\}$ and $N(x, U) = \{n \in \mathbb{Z}_+ : f^n(x) \in U\}$. The map f is said to be *topologically transitive* if for any pair of nonempty open subsets U, V of X, the set $N(U, V)$ is nonempty, *totally transitive* if each iterate f^k, $k \in \mathbb{N}$, is topologically transitive, *weakly mixing* if $f \times f$ is topologically transitive and *topologically mixing* if for any pair of nonempty open subsets U, V of X, there exists $N \in \mathbb{N}$ such that $n \in N(U, V)$ for all $n \geq N$. It is well known that topological mixing ⇒ weak mixing ⇒ total transitivity ⇒ topological transitivity [3]. Note that if f has the shadowing property, then topological transitivity coincides with chain transitivity and topological mixing coincides with chain mixing. The map f is said to have the *specification property* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \in \mathbb{N} \setminus \{1\}$, any finite sequence $y_1, y_2, \ldots, y_n \in X$ and any sequence $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ of natural numbers with $a_{i+1} - b_i \geqslant N$ for $i = 1, 2, \ldots, n-1$, there is a point $z \in X$ such that $d(f^j(z), f^j(y_k)) < \epsilon$ for every k with $1 \leq k \leq n$ and every j with $a_k \leq j \leq b_k$.

For any $A \subseteq \mathbb{Z}_+$, we define the *upper density* of A by

$$u_d(A) = \limsup_{n\to\infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n-1\}|,$$

where $|C|$ denotes the cardinality of the set $C \subseteq \mathbb{Z}_+$. A set $A \subseteq \mathbb{Z}_+$ has positive upper density if $u_d(A) > 0$. The map f is said to be *topologically ergodic* if for any pair of nonempty open subsets U, V of X we have $u_d(N(U, V)) > 0$ and it is said to be *totally topologically ergodic* if each f^k, $k \in \mathbb{N}$, is topologically ergodic. A set $A \subseteq \mathbb{Z}_+$ is said to be *syndetic* if it has bounded gaps, that is, there exists $N \in \mathbb{N}$ such that $[n, n+N] \cap A \neq \emptyset$ for every $n \in \mathbb{Z}_+$. The map f is said to be *strongly ergodic* if for any pair of nonempty open subsets U, V of X, the set $N(U, V)$ is syndetic and *totally strongly ergodic* if each f^k, $k \in \mathbb{N}$, is strongly ergodic. A point $x \in X$ is said to be *minimal* if $N(x, U)$ is syndetic for every neighbourhood U of x. The set of minimal points of f is denoted by $M(f)$. Note that if (X, f) is a dynamical system, then $M(f) = M(f^k)$ for every $k \in \mathbb{N}$. A nonempty, closed, f-invariant subset F of X is said to be *minimal* if it has no nonempty, proper, closed, f-invariant subset.

We also recall the *asymptotic* and *proximal* relations which are given by $Asym(f) = \{(x, y) \in X \times X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$ and $Prox(f) = \{(x, y) \in X \times X : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$. Elements in $Asym(f)$ and $Prox(f)$ are called asymptotic and proximal pairs of f respectively. Clearly, $Asym(f) \subseteq Prox(f)$. The map f is said to be *distal* if $Prox(f) = \Delta_X$, where $\Delta_X = \{(x, x) : x \in X\}$ is the diagonal of $X \times X$ and it is said to be *almost distal* if $Prox(f) \setminus Asym(f) = \emptyset$. It is clear that distal ⇒ almost distal.

3. The ALASP and ergodicity

Lemma 3.1 [7]**.** *Let (X, f) be a dynamical system and f be surjective. If f has the ALASP, then f is chain transitive.*

Theorem 3.2. *Let (X, f) be a dynamical system. If f is a surjection having the ALASP and the shadowing property, then $\overline{M(f)} = X$.*

Proof. Let $U \subseteq X$ be a nonempty open set and $u \in U$. Choose $\epsilon > 0$ such that $B(u, \epsilon) \subseteq U$, where $B(u, \epsilon) = \{y \in X : d(y, u) < \epsilon\}$. Suppose δ, $0 < \delta < \epsilon$, is obtained for $\epsilon/4$ and $\eta > 0$ is obtained for $\delta/2$ by the shadowing property of f. By Lemma 3.1, there exists an η-chain, say $\{u_0 = u, u_1, u_2, \ldots, u_k = u\}$, of f from u to u. For $m \in \mathbb{Z}_+$, define

$$w_{mk+j} = u_j, \quad 0 \leqslant j \leqslant k-1,$$

then $\{w_i\}_{i \geqslant 0} \equiv \{u_0 = u, u_1, \ldots, u_{k-1}, u_0 = u, u_1, \ldots, u_{k-1}, \ldots\}$ is an η-pseudo-orbit of f. So there exists $x \in X$ such that $d(f^i(x), w_i) <$