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Fourier transforms on Cantor sets: A study in non-Diophantine arithmetic and calculus

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ABSTRACT

Fractals equipped with intrinsic arithmetic lead to a natural definition of differentiation, integration, and complex structure. Applying the formalism to the problem of a Fourier transform on fractals we show that the resulting transform has all the required basic properties. As an example we discuss a sawtooth signal on the ternary middle-third Cantor set. The formalism works also for fractals that are not self-similar.

1. Introduction

While trying to formulate quantum mechanics on fractal backgrounds, one immediately faces the problem of momentum representation. The issue is nontrivial and reduces to the question of what should be meant by a Fourier transform on a fractal. Historically the first approach to fractal harmonic analysis can be, implicitly, traced back to studies of diffusion on fractals [1,2]. A generator of the diffusion is then a candidate for a Laplacion on a fractal, and once we have a Laplacian we can look for its eigenfunctions. The eigenfunctions may play a role of a Schauder basis in certain function spaces, and thus lead to a sort of signal analysis on a fractal. Whether and under what conditions the resulting eigenfunction expansions can be regarded as analogs of Fourier transformations is a separate story. Fractals such as Cantor sets naturally lead to wavelet transforms (the Haar basis [3–5], for example), but quantum mechanical momentum representation is expected to be associated with gradient operators, and there is no obvious link between Haar wavelets and gradients.

Gradients and Laplacians can be defined on fractals also more directly. Here one should mention the approaches that begin with Dirichlet forms defined on certain self-similar fractals, and those that start with discrete Laplacians [6–9]. Self-similarity is an important technical assumption, and it is not clear what to do in more realistic cases, such as multi-fractals or fractals that have no self-similarity at all (a generic case in natural systems). Four different definitions of a gradient (due to Kusuoka, Kigami, Strichartz and Teplyaev) can be found in [10].

One might naively expect that it would be more logical to begin with first derivatives and only then turn to higher-order operators, such as Laplacians. It turns out that Laplacians defined in the above ways cannot be regarded as second-order operators. Still, an approach where Laplacians are indeed second-order is possible and was introduced by Fujita [11,12], and further developed by Freiberg, Zähle and others [13–17]. We will later see that a non-Diophantine Laplacian is exactly second-order and, similarly to the approach from [13–17], is based on derivatives and integrals satisfying the fundamental laws of calculus.

In yet another traditional approach to harmonic analysis on fractals, one begins with self-similar fractal measures, and then seeks exponential functions that are orthogonal and complete with respect to them. The classic result of Jorgensen and Pedersen [18] states that such exponential functions do exist on certain fractals, such as the quaternary Cantor set, but cannot be constructed in the important case of the ternary middle-third Cantor set.

In the present paper, we will follow a different approach. One begins with arithmetic operations (addition, subtraction, multiplication, and division) which are intrinsic to the fractal. The arithmetic so defined is non-Diophantine in the sense of Burgin [19,20]. An important step is then to switch from arithmetic to calculus [21] where, in particular, derivatives and integrals are naturally defined. The resulting formalism is simple and general, extends beyond fractal applications, but works with no difficulty for Cantorian fractals, even if they are not self-similar [21,22]. Actually, a straightforward motivation for the present paper came from discussions with the referee of [22], who pointed out

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possible difficulties with momentum representation of quantum mechanics on Cantorian space-times. In the sequel to the present paper [27], we show how to generalize the construction to fractals of a Sierpiński type.

In Section 2 we recall the basic properties of non-Diophantine arithmetic, illustrated by four examples from physics, cognitive science, and fractal theory. Section 3 is devoted to complex numbers, discussed along the lines proposed by one of us in [21], and with particular emphasis on trigonometric and exponential functions. In Section 4 we recall the non-Diophantine-arithmetic definitions of derivatives and integrals. Section 5 discusses a scalar product of functions, and the corresponding Fourier transform (both complex and real) is introduced in Section 7. In Section 8 we discuss an explicit example of a sawtooth signal with Cantorian domain and range. Finally, in Section 9 we briefly discuss the issue of spectrum of Fourier frequencies, and compare our results with those from [18].

2. Generalized arithmetic: Fractal and not only

Consider a set $\mathbb{X}$ and a bijection $f:\mathbb{X}\to\mathbb{R}$ Following the general formalism from [21] we define the arithmetic operations in $\mathbb{X}$,

$$x\oplus y = f^{-1}\big(f(x)+f(y)\big),$$
$$x\ominus y = f^{-1}\big(f(x)-f(y)\big),$$
$$x\odot y = f^{-1}\big(f(x)f(y)\big),$$
$$x\oslash y = f^{-1}\big(f(x)/f(y)\big),$$

for any $x, y\in\mathbb{X}$. In later applications we will basically concentrate on an appropriately constructed fractal $\mathbb{X}$, but the results are more general. This is an example of a non-Diophantine arithmetic [19,20].

One verifies the standard properties: (1) associativity $(x\oplus y)\oplus z = x\oplus(y\oplus z)$, $(x\odot y)\odot z = x\odot(y\odot z)$, (2) commutativity $x\oplus y = y\oplus x$, $x\odot y = y\odot x$, (3) distributivity $(x\oplus y)\odot z = (x\odot z)\oplus(y\odot z)$. Elements $0', 1'\in\mathbb{X}$ are defined by $0'\oplus x = x$, $1'\odot x = x$, which implies $f(0') = 0$, $f(1') = 1$. One further finds $x\ominus x = 0'$, $x\oslash x = 1'$, as expected. A negative of $x\in\mathbb{X}$ is defined as $\ominus x = 0'\ominus x = f^{-1}\big(-f(x)\big)$, i.e. $f(\ominus x) = -f(x)$ and $f(\ominus 1') = -f(1') = -1$, i.e. $\ominus 1' = f^{-1}(-1)$. Notice that

$$(\ominus 1')\odot(\ominus 1') = f^{-1}\big(f(\ominus 1')^2\big) = f^{-1}(1) = 1'. \tag{1}$$

Multiplication can be regarded as repeated addition in the following sense. Let $n\in\mathbb{N}$ and $n' = f^{-1}(n)\in\mathbb{X}$. Then

$$n'\oplus m' = (n+m)', \tag{2}$$

$$n'\odot m' = (nm)' \tag{3}$$

$$= \underbrace{m'\oplus\cdots\oplus m'}_{n\text{times}}. \tag{4}$$

In particular $n' = 1'\oplus\cdots\oplus 1'$ (n times).

A power function $A(x) = x\odot\cdots\odot x$ (n times) will be denoted by $x^{n'}$. Such a notation is consistent in the sense that

$$x^{n'}\odot x^{m'} = x^{(n+m)'} = x^{n'\oplus m'}. \tag{5}$$

Before we plunge into fractal applications let us consider four explicit examples of non-Diophantine arithmetic.

2.1. Benioff's number scaling

The number-scaling approach of Benioff [23,24] can be regarded as a particular case of the above formalism with $f(x) = px$, $p\neq 0$. Indeed, $x\odot y = (1/p)(pxpy) = pxy$, $x\oplus y = (1/p)(px+py) = x+y$, $x\oslash y = (1/p)(px)/(py) = x/(py)$, but $f(1/p) = 1$. Since $(1/p)\odot x = (1/p)\big(p(1/p)px\big) = x$ one infers that $1' = f^{-1}(1) = 1/p$ is the unit element of multiplication in Benioff's non-Diophantine arithmetic.

2.2. Fechner map

This arithmetic is implicitly used in cognitive science [25]. It occurs as a solution of the following Weber–Fechner problem [26]: Find a generalized arithmetic such that $(x+kx)\ominus x$ is independent of x. Here $x\mapsto x' = x+\Delta x$ is the change of an input signal, while $x'\ominus x$ is the change of x as perceived by a nervous system. Experiments show that $\Delta x/x\approx k = \text{const}$ (Weber-Fechner law) in a wide range of xs, and with different values of k for different types of stimuli. The corresponding arithmetic is defined by the 'Fechner map' $f(x) = a\ln x + b$, $f^{-1}(x) = e^{(x-b)/a}$, and thus $0' = f^{-1}(0) = e^{-b/a}$, $1' = f^{-1}(1) = e^{(1-b)/a}$. Clearly, $0'\neq 0$ and $1'\neq 1$. Interestingly, the Fechnerian negative of $x\in\mathbb{R}_+$ reads

$$\ominus x = 0'\ominus x = e^{-2b/a}/x\in\mathbb{R}_+, \tag{6}$$

but nevertheless does satisfy

$$\ominus x\oplus x = e^{-b/a} = 0', \tag{7}$$

as it should on general grounds [25]. So, numbers that are negative with respect to one arithmetic are positive with respect to another. In a future work we will show that Fechner's f has intriguing consequences for relativistic physics.

2.3. Ternary Cantor line

Let us start with the right-open interval $[0, 1)\subset\mathbb{R}$, and let the (countable) set $\mathbb{Y}_2\subset[0, 1)$ consist of those numbers that have two different binary representations. Denote by $0.t_1t_2\dots$ a ternary representation of some $x\in[0, 1)$. If $y\in\mathbb{Y}_1 = [0, 1)\setminus\mathbb{Y}_2$ then y has a unique binary representation, say $y = 0.b_1b_2\dots$. One then sets $g_\pm(y) = 0.t_1t_2\dots$, $t_j = 2b_j$. The index $\pm$ appears for the following reason. Let $y = 0.b_1b_2\dots = 0.b'_1b'_2\dots$ be the two representations of $y\in\mathbb{Y}_2$. There are two options, so we define: $g_-(y) = \min\{0.t_1t_2\dots, 0.t'_1t'_2\dots\}$ and $g_+(y) = \max\{0.t_1t_2\dots, 0.t'_1t'_2\dots\}$, where $t_j = 2b_j$, $t'_j = 2b'_j$. We have therefore constructed two injective maps $g_\pm$: $[0, 1)\to[0, 1)$. The ternary Cantor-like sets are defined as the images $C_\pm(0, 1) = g_\pm\big([0, 1)\big)$, and $f_\pm$: $C_\pm(0, 1)\to[0, 1)$, $f_\pm = g_\pm^{-1}$, is a bijection between $C_\pm(0, 1)$ and the interval. For example, $1/2\in\mathbb{Y}_2$ since $1/2 = 0.1_2 = 0.0(1)_2$. We find

$$g_-(1/2) = \min\{0.2_3 = 2/3, 0.0(2)_3 = 1/3\} = 1/3, \tag{8}$$

$$g_+(1/2) = \max\{0.2_3 = 2/3, 0.0(2)_3 = 1/3\} = 2/3. \tag{9}$$

Accordingly, $1/3\in C_-(0, 1)$ while $2/3\notin C_-(0, 1)$. And vice versa, $1/3\notin C_+(0, 1)$, $2/3\in C_+(0, 1)$. The standard Cantor set is the sum $\tilde{C} = C_-(0, 1)\cup C_+(0, 1)$. All irrational elements of $\tilde{C}$ belong to $C_\pm(0, 1)$ (an irrational number has a unique binary form), so $\tilde{C}$ and $C_\pm(0, 1)$ differ on a countable set. Notice further that $0\in C_\pm(0, 1)$, with $f_\pm(0) = 0$. In [21,22] we worked with $C_-(0, 1)$ so let us concentrate on this case. Let $C_-(k, k+1)$, $k\in\mathbb{Z}$, be the copy of $C_-(0, 1)$ but shifted by k. We construct a fractal $\mathbb{X} = \cup_{k\in\mathbb{Z}}C_-(k, k+1)$, and the bijection $f:\mathbb{X}\to\mathbb{R}$. Explicitly, if $x\in C_-(0, 1)$, then $x+k\in C_-(k, k+1)$, and $f(x+k) = f(x)+k$ by definition. In [21,22] the set $\mathbb{X}$ is termed the Cantor line, and f is the Cantor-line function. For more details see [21]. The set $\mathbb{X}\cap[k, k+1)$ is self-similar, but $\mathbb{X}$ as a whole is not-self similar. Fig. 1 (upper) shows the plot of $g = f^{-1}$. For completely irregular generalizations of the Cantor line, see [22].

Let us make a remark that in the literature one typically considers Cantor sets $\tilde{C}$ so that the resulting function $g:\tilde{C}\to[0, 1)$ is non invertible on a countable subset. In [3] one employs the map