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Spectral representations and global maps of cellular automata dynamics

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ABSTRACT

We present a spectral representation of any computation performed by a Cellular Automaton (CA) of arbitrary topology and dimensionality via an appropriate coding scheme in Fourier space that can be implemented in an analog machine ideally circumventing part of the overall waste heat production. We explore further consequences of this encoding and we provide a simple example based on the "Game-of-Life" where we find global maps for small lattices indicating an interesting underlying recursive structure.

1. Introduction

Evolution of ALife has more than 50 years of a rich history from the first attempts of Von Neumann and Barricelli towards self-reproducing and evolving digital organisms to nowadays attempts for whole world sentient simulations. The field has recently been reviewed by Taylor in [1,2]. The particular class of simulation tools known as Cellular Automata (CA) were first introduced by Von Neumann [3] and later popularized by Conway [4] and Wolfram [5,6]. The latter in fact represents a turn towards a rather strong conjecture in favor of the so called *Church-Turing Thesis* [7] for all of reality including perhaps biological entities both natural and synthetic as being effectively computable. Philosophical ramifications aside, it appears that for all practical purposes certain aspects of reality are well captured in various circumstances by similar discrete models which stand for good alternatives in comparison with the numerically heavy and time consuming ones based on differential equations both ordinary and partial. Well known examples include diffusion-reaction models, fire and disease spread models, lattice-gas automata and so on.

The scope of the present article is not in altering anything in the original definitions of all similar classes of automata which are self-sufficient but rather to complement their study which in many respects remains difficult due to their combinatory nature as it often happens with problems in discrete mathematics. The possibility of a complete analog implementation of both parallel and serial machines outside the ordinary Von Neumann architecture of stack-based machines will be explored in subsequent publications. In the course of developing this approach we also recover an equivalence that allows mapping a general class of discrete complex systems into a more compact form which is a subset of signal theory that does not appear in the present literature and is worth reporting independently of the other objectives. This is inherently linked to the primary objective in that a complete transcription of states into signals in a particular way, as explained in the next sections, is necessary for such a feat.

Motivation behind the main objective is based on two serious reasons of an entirely practical as much as pressing nature that are now attracting wide attention. The first is the expected breakdown of Moore's law and the need for alternative computing machinery. It is the author's opinion that this heralds also a need for reviewing analog computing, not through the old Leibnizian paradigm, but the one followed in Shannon's original 1945 GPAC (General Purpose Analog Computer) [8].

The second reason is the need to reduce if not entirely eliminate the tremendous waste heat produced by the constantly expanding data center infrastructure. This is a mostly serious problem in the context of global warming and overburdening of current electrical power plants for cooling which cannot be overcome without extreme measures as the recent announcement by Google on the need to install its next such centers underwater. Even if heat recycling and other techniques were to be used, the current state of the art as well as the extreme difficulties in advancing true quantum based computing infrastructure justifies

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the search for alternative computing techniques that could be both less costly and more efficient. As any and all of the above reasons represent true limits to growth and thus potential evolutionary threats, the paradigm presented here could be expanded to many possible directions, equally beneficial.

The simplest model of parallel computing with synchronous automata given by the CA paradigm is here utilized to present a generic transcription method that can be realized either with existing Digital Signal Processing (DSP) or purely analog techniques. The already proven computational universality for a particular subset of even elementary CA provides sufficient motivation for their use in the particular transcription method applied here. It will be subsequently shown that any such device is effectively equivalent to a set of nonlinearly coupled filters.

In the effort to unify their study and also allow for arbitrary neighborhood topologies with higher dimensionalities in an easily understandable and more efficient way that avoids some unnecessary obfuscation of symbol tables and similar, we introduce a compact treatment which allows an analytical reformulation as a set of dynamical systems in a rather transparent way. To this aim, we present a different encoding of the dynamics as a composite of a linear and a nonlinear mapping and we concentrate on certain mathematical aspects of them that appear to be universal over a very broad class of models which may extend beyond elementary CA, independently of their particular scope.

It is equally possible to perform a similar transcription for serial machines but the process is lengthier and requires additional measures. This research is currently underway and it will be presented in a subsequent publication. Hopefully, this will also allow more advanced treatment by other researchers using the full armory of dynamical systems theory in uncovering important aspects of the dynamics in a variety of different practical situations including social simulations and artificial life studies.

In the next sections we present the necessary definitions from convolution calculus for understanding the tools developed. In Section 2 we present the basic machinery behind the decomposition. In Section 3 we also show an additional possibility an alternative type of dynamics appropriate for analog machines with special encoding which guarantees constant entropy during evolution of the set of all memory states. In Section 4 we comment on the technical details for expanding the previous description for arbitrary dimensionality and in Section 5 we use a 2D CA with a rule for the game of life to provide a toolbox for experimentation on this new technique and some results on global maps that can be easily obtained in the given framework.

2. CA dynamics in a dual Fourier space

While the original definition of CA is relying heavily on old nomenclature invented in the 30s by mathematicians like Turing and Post, for the general theory of Automata, the case of synchronous CA can be treated with relative ease by introducing certain tools from linear algebra and convolution calculus. We remind that ordinary CA are defined by a tuple $\{L^D, \sigma, N, T\}$ where L stands for the length of a hypercubic integer lattice of dimension D, N stands for a neighborhood of arbitrary topology around each cell on the lattice taking values from an alphabet σ in some base b arithmetic, T is a transition function usually defined as a look-up table (LUT) encoding all possible combinations of cell states of any neighborhood and the associated next states of the central cell. Ordinary symmetric neighborhoods are special cases in the general set N defined by a radius r with a width $2r+1$ giving rise to 2^{2r+1} neighborhood combinations in the 1D case and similarly for higher dimensions. A set of appropriate boundary conditions is most often taken on a toroidal, multiply periodic topology which we also use here.

As a first step we recognize that any transition table can also be redefined as a 1D graph of a discrete transfer function. This is due to the fact that any symbol-wise function of many discrete variables on some bounded space is always reducible into an equivalent 1D form by taking the polynomial representation of its multi-index space as a mapping to a unique index. Thus from any such LUT of the form $T(c_1, \dots, c_k) \to c', \{c_n\}_{n=1}^{k}, c' \in N$ mapping neighborhood values into new cell states, one can always derive a single transfer function graph through a complete enumeration of all combinations. Given an alphabet σ in base b arithmetic any such map can always be written analytically in the form

$$T \to f : N \times \sigma \to \sigma : c' = f\left(c_1 + c_2 b + \dots + c_k b^{k-1}\right) \tag{1}$$

for any and all of the b^k possible rules given as symbol strings. In the above we made use of the polynomial representation for mapping all possible neighbor combinations into the set of integers in the interval $[0, \dots, b^k - 1]$.

Before analyzing the role of higher dimensionality for the lattice itself, we restrict attention to the case of a 1D CA. It is then possible to utilize a particular addressing scheme for (1) which leads directly to a decomposition of the dynamics in two separate maps as $L_{n+1} = (f \circ g)(L_n)$ where f stands for the generally nonlinear part and g is a strictly linear map. We do this by introducing a generic *Interaction Kernel* as a special $L \times L$ circulant matrix of the form

$$\mathbf{C} = \begin{bmatrix} s_1 b & s_2 b^2 & \dots & s_0 \\ s_0 & s_1 b & \cdots & s_{L-1} b^{L-1} \\ \vdots & & \ddots & \vdots \\ s_2 b^2 & \cdots & s_0 & s_1 b \end{bmatrix} \tag{2}$$

Any such matrix is composed from the cyclic permutations of the vector $\mathbf{c} = [s_0, s_1 b, \dots, s_{L-1} b^{L-1}]$ where we use the binary symbols s_i to denote the topology of any possible neighborhood via a logical mask plus a wraparound at the matrix edges for the cyclic boundary conditions. Any such matrix is a special case of a Toeplitz matrix and it is fairly easy to realize it from a single vector in the first row henceforth referred as the "kernel vector" wherever a generic routine for constructing Toeplitz matrices is available. In Matlab, the unique command *toeplitz* (**c**(1), fliplr (**c**(2:end)), **c**) is sufficient yet, even this is not necessary. As we will show, a particularly elegant solution is offered by the general diagonal form from which one deduces that only the DFT over the defining first row is actually necessary and this technique was used in some of the example codes. This is a ubiquitous technique in modern signal processing and image analysis and it is further discussed in the case of higher dimensional automata in Section 4.

We notice the existence of two important limits which we will term for convenience the "*Holographic*" and "*Anti-holographic*" limits with the first corresponding to the case of the binary mask digits all ones (global coupling) and the second to the minimal case of only 1st order nearest neighbors. The binary mask allows even for random, disconnected ("non-local") neighborhoods. In such a case the C matrix will obtain a sparse band structure. In the most trivial case of nearest neighbors apparently one only has a sparse matrix of a single 3 entries band around the main diagonal.

The above definitions allow decomposing the dynamics of any 1D CA as a set of two discrete evolution equations in the form

$$\begin{aligned} h_n &= \mathbf{C} \cdot L_n \\ L_{n+1} &= f(h_n) \end{aligned} \tag{3}$$

We now note that the above picture coincides with that of a special class of filters with a circulant connectivity matrix of weights given by the kernel C and h playing the role of what is known in neural network terminology as the "excitation field". The new variable h stands for an intermediate addressing field which is passed through the transfer function with the method