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Convection induced by instabilities in the presence of a transverse seepage

Takeshi Akinaga [a,c], Tomoaki Itano [b], Sotos Generalis [c,*]

[a] Marie Sklodowska Curie International Incoming Fellow, United Kingdom
[b] Department of Pure and Applied Physics, Faculty of Engineering Science, Kansai University, Osaka 564-8680, Japan
[c] System Analytics Research Institute, Aston University, Birmingham, B4 7ET, United Kingdom

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ABSTRACT

The transition of laterally heated flows in a vertical layer and in the presence of a streamwise pressure gradient is examined numerically for the case of different values Prandtl number. The stability analysis of the basic flow for the pure hydrodynamic case ($Pr = 0$) was reported in [1]. We find that in the absence of transverse pumping the previously known critical parameters are recovered [2], while as the strength of the Poiseuille flow component is increased the convective motion is delayed considerably. Following the linear stability analysis for the vertical channel flow our attention is focused on a study of the finite amplitude secondary travelling-wave (TW) solutions that develop from the perturbations of the transverse roll type imposed on the basic flow and temperature profiles. The linear stability of the secondary TWs against three-dimensional perturbations is also examined and it is shown that the bifurcating tertiary flows are phase-locked to the secondary TWs.

1. Introduction and motivation

Theoretical investigations aided by the advance of powerful hardware and parallel experimental studies of the stability of plane parallel shear flows have provided significant insights in identifying the mechanisms of instability and transition from the laminar to the turbulent state of shear fluid flow, via the approach based on the sequence of bifurcations. Parallel shear flows include the well studied plane Couette flow, plane Poiseuille flow (PPF), homogeneously heated flow, laterally heated flow (LHF) and Rayleigh–Bénard (natural) convection. In [2] the parallel shear flow LHF was studied for the case of $Pr = 0$ without the imposition of a constant pressure gradient was presented. In [1] the linear stability for the $Pr = 0$ case was examined for a wide range of the Grashof and Reynolds numbers in order to extract only the fluid dynamic instability mechanisms. The present work complements and extends the work of [1] by examining the nonlinear flow that emerges at the stability boundary of the basic flow configuration. The general form of the Navier–Stokes equations can be expressed by $\frac{\partial \mathbf{\Phi}}{\partial t} = \boldsymbol{M}(\boldsymbol{R})\ \mathbf{\Phi} + \boldsymbol{N}(\mathbf{\Phi}, \mathbf{\Phi})$. In this set of coupled partial differential equations, $\mathbf{\Phi}$ describes the state of the system, $\boldsymbol{M}(\boldsymbol{R})$ is a linear and $\boldsymbol{N}(\cdot, \cdot)$ is a nonlinear operator that involve partial derivatives. $\boldsymbol{R}$ represents collectively the parameters of the system, such as the angle of inclination of the channel flow, the Prandtl, the Rayleigh (or Grashof), the Reynolds and the wave numbers. In our sequential bifurcation approach we first obtain the stability of the basic state $\mathbf{\Phi}_0$, which satisfies:

$$0 = \boldsymbol{M}(\boldsymbol{R})\mathbf{\Phi}_0 + \boldsymbol{N}(\mathbf{\Phi}_0, \mathbf{\Phi}_0),$$

via the introduction of infinitesimal disturbances, $\widetilde{\mathbf{\Phi}}$ and by ignoring the nonlinear terms:

$$\frac{\partial \widetilde{\mathbf{\Phi}}_0}{\partial t} = \boldsymbol{M}(\boldsymbol{R})\widetilde{\mathbf{\Phi}}_0 + \boldsymbol{N}(\mathbf{\Phi}_0, \widetilde{\mathbf{\Phi}}_0)\ +\ \boldsymbol{N}(\widetilde{\mathbf{\Phi}}_0, \mathbf{\Phi}_0). \tag{1}$$

The nonlinear states grow from the stability boundary of the basic state, that is typically characterised by $f_0(\boldsymbol{R}) = 0$, where f_0 is the neutral surface of the basic state in parameter space $\boldsymbol{R}$. If we repeat the aforementioned procedure of perturbation on the obtained nonlinear states, we could find a $f_i(\boldsymbol{R}) = 0$, where $f_i, i = (1, 2, \ldots)$, is the corresponding neutral surface of the nonlinear state, thus defining a successive sequence of bifurcations for the higher order states en route to turbulence.

Motivated by the desire to understand the transition from laminar flow to the turbulent state for this (mixed convection) problem, we revisited the pure hydrodynamic case, first studied in [1] through to tertiary level, and also extended it to the case $Pr = 0.71$. In the following section we formulate the problem and in Section 3 we show the numerical method that we employ to investigate the linear stability of our basic steady state.

* Corresponding author.
E-mail address: s.c.generalis@aston.ac.uk (S. Generalis).

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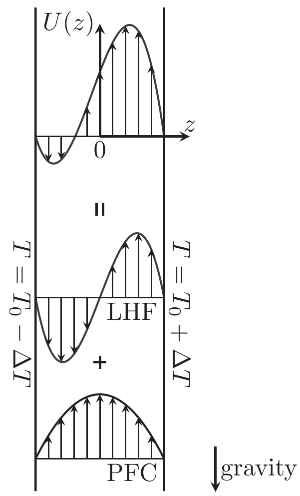

Fig. 1. Geometrical configuration of the problem. In this figure $-d \le z \le d$.

In Section 4 we study the nonlinear states that are generated at the stability boundary of the basic state of our problem. In Section 5 we study the stability of the bifurcated secondary flow and we therefore found the manifold of points where the tertiary states are generated. In Section 6 we provide our conclusions and we identify avenues for future research.

2. Mathematical formulation and primary stability

We consider an incompressible Boussinesq Newtonian fluid bounded by two vertical parallel plates of infinite extent with different temperatures $T_0 + \Delta T$ and $T_0 - \Delta T$, subject to a pressure gradient deviation from the hydrostatic pressure, see Fig. 1. The origin of the Cartesian coordinates system is positioned in the mid-plane of the fluid layer of width 2d, taking x, y, z coordinates as the streamwise, spanwise and wall normal directions with unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$. We assume here the state satisfies the periodic boundary conditions with wave numbers α and β for the x and y directions, respectively. For the non-dimensional form of the velocity vector, the temperature and pressure deviations from the basic state as $\boldsymbol{u}$, θ and π, we obtain the non-dimensional form of the equations of motion in the following form:

$$\frac{\partial}{\partial t}\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\,\Pi + \theta\boldsymbol{i} + \nabla^2\boldsymbol{u}, \tag{2}$$

$$\frac{\partial}{\partial t}\theta + \boldsymbol{u}\cdot\nabla\theta = Pr^{-1}\nabla^2\theta, \tag{3}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{4}$$

The Boussinesq approximation is used in that the material properties are assumed to be constant except for the linear temperature dependence of the density, which has been taken into account in the buoyancy term only. The physical properties of the system are characterised by the three non-dimensional parameters $Gr = g\gamma\Delta T d^3/\nu^2$, the Grashof number that gives the strength of the heating, $R = U_{\max} d/\nu$, the Reynolds number that measures the strength of the applied pressure gradient in the streamwise direction ($U_{\max}$ is the speed of Poiseuille laminar flow at the origin of the coordinate system) and the Prandtl number $Pr = \nu/\kappa$. Here κ is the thermal diffusivity, ν is the kinematic viscosity, γ is the coefficient of thermal expansion and g is the acceleration due to gravity. For the non-dimensional description of the problem we employ d, d^2/ν and $\Delta T/PrGr$, as the units of length, time and temperature, respectively. Eqs. (2–4) are supplemented by the boundary conditions:

$$\boldsymbol{u} = \theta = 0 \text{ at } z = \pm 1. \tag{5}$$

The basic flows are given by:

$$U_b(z) = -\frac{Gr}{6}(z^3 - z) + Re(1 - z^2), \quad T_b(z) = \; Gr\, z \quad \text{for} \quad |z| \le 1. \tag{6}$$

In order to be able to identify other, than the laminar, solutions for Eqs. (2–4) it is convenient to introduce a general representation for solenoidal vector fields [3] and to write:

$$\boldsymbol{u} = \check{U}\boldsymbol{i} + \nabla\times(\nabla\times\phi\boldsymbol{k}) + \nabla\times\psi\boldsymbol{k} \tag{7}$$

where the poloidal and toroidal potentials, ϕ, ψ, respectively, are uniquely defined if their averages over the planes z = constant vanishes. The mean flow and mean temperature $\check{U}$, $\check{T}$, respectively, are given by:

$$\partial_z^2\check{U} + \check{T} + \partial_z\overline{\Delta_2\phi(\partial_x\partial_z\phi + \partial_y\psi)} = \partial_t\check{U}, \tag{8}$$

$$\partial_z^2\check{T} + Pr\partial_z\overline{\Delta_2\phi\theta} = Pr\partial_t\check{T}, \tag{9}$$

where the overbar denotes an average over the planes $z = constant(s)$, and Δ_2 denotes the two dimensional Laplacian, $\Delta_2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. Since we anticipate a vanishing mean flow in the y-direction, we have neglected it in expression (7). Eqs. (8) and (9) have been obtained by taking the x, y-average of Eqs. (2) and (3) and by subtracting the basic solution $U_0\boldsymbol{i}$. In order to obtain transverse vortices we use the Fourier expansions for the variables (ϕ, ψ, θ) in $\boldsymbol{\Phi} = (\boldsymbol{u}, \theta) = (u, v, w, \theta)$, see also [2,6–8]:

$$\phi(x,y,z,t) = \sum_{n=0}^{N}\sum_{\substack{m=-M \\ (m,l)\neq(0,0)}}^{M}\sum_{l=-L}^{L} a_{nml}\left(1-z^2\right)^2\, T_n(z) \times \exp(\mathrm{i}m\alpha(x-ct) + \mathrm{i}\; l\,\beta y),$$
$$\theta(x,z,t) = \sum_{n=0}^{N}\sum_{\substack{m=-M \\ (m,l)\neq(0,0)}}^{M}\sum_{l=-L}^{L} b_{nml}\left(1-z^2\right)\, T_n(z) \times \exp(\mathrm{i}m\alpha(x-ct) + \mathrm{i}\; l\,\beta y), \tag{10}$$

while we write:

$$\check{U} = \sum_{n=0}^{N} C_n(1-z^2)T_n(z), \quad \check{T} = \sum_{n=0}^{N} D_n(1-z^2)T_n(z), \tag{11}$$

where N, M and L are the truncation levels for the complex coefficients a_{nml}, and $T_n(z)$ is the nth order Chebyschev polynomial (Fig. 6). We have incorporated the phase velocity c in the expansion, so that calculations can be performed on a moving frame that is phase locked with the nonlinear solutions bifurcating from the neutral curve boundaries. The factor $(1-z^2)^2$ has been inserted in the expression for ψ in order to take into account the boundary conditions expressed by Eq. (5). By substituting the expansions (10) into the streamwise projection of the curl and double curl of (2), rewriting (3) using (7), and finally multiplying the resulting equations by $\int_0^{2\pi/\alpha} dx \exp \mathrm{i}l\,\alpha\, x$, $l = 1, 2, \ldots,$ we can evaluate the expansions (10) at each collocation point, i.e. we obtain a set of nonlinear algebraic equations for the complex coefficients a_{nm}. Further details of the numerical method employed here to obtain nonlinear solutions have been presented recently in [6] and [7,8] in relation to different types of shear flows. We elaborate on the results of our simulations in the next section. For the numerical solution of the resulting systems of equations a truncation scheme must be introduced. The boundaries of Fig. 2 and the numerical values of Table 1 are obtained by setting $m = 1$, $l = 0$ in Eq. (10) and by