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Resonant functional problems of fractional order

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ABSTRACT

We study a resonant functional boundary value problem of fractional order. Our results are based on the coincidence degree theory of Mawhin and improve those in a recent paper of Y. Zou and Y. Cui [Adv. Difference Equ. 2013 (2013), 1–25].

1. Introduction

Fractional differential equations arise in a variety of different areas such as rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, etc. We refer the reader to the monographs [3,6–8] for the theory and applications of fractional calculus. There are many results for the study of fractional boundary value problems, see [1,2,4,9,11]. In [11], Zou and Cui investigated the existence of solutions for the problem

$$D^{\alpha}_{0^+}u(t) = f(t, u(t), D^{\alpha-1}_{0^+}u(t), D^{\alpha-2}_{0^+}u(t)), \quad \text{a.e. } t \in [0,1], \tag{1}$$

$$I^{3-\alpha}_{0^+}u(0) = 0, \quad \Phi_1[D^{\alpha-1}_{0^+}u(t)] = \Phi_2[D^{\alpha-2}_{0^+}u(t)] = 0, \tag{2}$$

where $2 < \alpha < 3$, $\Phi_1, \Phi_2 : C[0,1] \to \mathbb{R}$ are continuous linear functionals. The authors considered the following sets of functional conditions:

(A_1) $\Phi_1(1)\Phi_2(1) \neq 0$;
(A_2) $\Phi_1(1) = \Phi_2(t) = 0,\ \Phi_2(1) \neq 0$;
(A_3) $\Phi_1(1) = \Phi_2(1) = 0,\ \Phi_2(t) \neq 0$;
(A_4) $\Phi_1(1) \neq 0,\ \Phi_2(1) = \Phi_2(t) = 0$;
(A_5) $\Phi_1(1) = \Phi_2(1) = \Phi_2(t) = 0$.

The assumption (A_1) corresponds to the non-resonance. The condition (A_2) defines a resonance with $\operatorname{Ker} L = \{at^{\alpha-1} : a \in \mathbb{R}\}$, and the conditions (A_3) and (A_4) are responsible for a resonance with $\operatorname{Ker} L = \{at^{\alpha-2} : a \in \mathbb{R}\}$. Finally, if (A_5) holds, then $\operatorname{Ker} L = \{at^{\alpha-2} + bt^{\alpha-1} : a, b \in \mathbb{R}\}$.

The results of [11] are obtained along the lines of [10].

However, the condition

$$\Phi_1(1) = 0, \quad \Phi_2(1), \Phi_2(t) \neq 0 \tag{3}$$

was not considered in which case $\operatorname{Ker} L = \{a(\Phi_2(1)t^{\alpha-1} - (\alpha-1)\Phi_2(t)t^{\alpha-2}) : a \in \mathbb{R}\}$. The existence theorems obtained in [11] do not apply to this important case.

In our paper, we prefer to consider a slightly different problem. Namely, we study

$$D^{\alpha}_{0^+}u(t) = f\Big(t, u(t), D^{\alpha-2}_{0^+}u(t), D^{\alpha-1}_{0^+}u(t)\Big), \quad t \in (0,1), \tag{4}$$

$$u(0) = 0, \quad B_1(u) = B_2(u) = 0, \tag{5}$$

of fractional order $2 < \alpha < 3$.

The outline of our paper is as follows. Firstly, we introduce the essentials of the Riemann-Liouville fractional integral and derivative as well as the coincidence degree theorem of Mawhin. In Section 1, the also formulate the abstract problem $Lu = Nu$ and, in Section 2, we state and prove the existence criteria. Throughout the paper, the reader may find a few remarks in which we point

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out the improvements and contributions to the method of the coincidence degree theory for fractional boundary value problems in general and, in particular, the extensions of [11].

The left side of (4) is the Riemann-Liouville fractional derivative defined in the general form and in terms of the Riemann-Liouville fractional integral by

$$\mathcal{D}_{0+}^{\kappa}u=\frac{d^n}{dt^n}\frac{1}{\Gamma(n-\kappa)}\int_0^t(t-s)^{n-\kappa-1}u(s)\,ds=\frac{d^n}{dt^n}\mathcal{I}_{0+}^{n-\kappa}u(t),$$

where $\kappa>0$, $n=[\kappa]+1$. The most important relations are the "inversion" properties, which the next two theorems (see, e. g. [8]) supply:

Theorem 1.1.

(a) *The equality* $\mathcal{D}_{0+}^{\kappa}\mathcal{I}_{0+}^{\kappa}y=y$ *holds for every* $\kappa>0$ *and* $y\in L_1(0,1)$*;*

(b) *For* $u\in L_1(0,1)$, $n=[\kappa]+1$, *if* $\mathcal{I}_{0+}^{n-\kappa}u\in AC^{n-1}[0,1]$, *then*

$$\mathcal{I}_{0+}^{\kappa}\mathcal{D}_{0+}^{\kappa}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{t^{\kappa-k-1}}{\Gamma(\kappa-k)}\left(\frac{d^{n-k-1}}{dt^{n-k-1}}\mathcal{I}_{0+}^{n-\kappa}u\right)(0).$$

For $\rho<0$, we introduce the notation $\mathcal{I}_{0+}^{\rho}=\mathcal{D}_{0+}^{-\rho}$.

Theorem 1.2. *If* $\beta,\rho+\beta>0$ *and* $y\in L_1(0,1)$, *then the equality*

$$\mathcal{I}_{0+}^{\rho}\mathcal{I}_{0+}^{\beta}y=\mathcal{I}_{0+}^{\rho+\beta}y$$

holds.

We work in Banach spaces

$$X=\{u:u,D_{0^+}^{\alpha-2}u,D_{0^+}^{\alpha-1}u\in C[0,1]\},\quad Y=L_1(0,1)$$

with the respective norms

$$\|u\|_X=\max\{\|u\|_0,\|D_{0^+}^{\alpha-2}u\|_0,\|D_{0^+}^{\alpha-1}u\|_0\}\ \text{and}\ \|y\|_1=\int_0^1|y(t)|\,dt,$$

where $\|u\|_0=\max_{t\in[0,1]}|u(t)|$.

Remark 1. It should be mentioned that in [11] a solution is sought in the space of functions

$$C^{\alpha-1}=\{u:u(t)=\mathcal{I}_{0+}^{\alpha-1}x(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2},\ x\in C[0,1]\}.$$

If $u\in C^{\alpha-1}$, then $u(0)=0$, which implies $\mathcal{I}_{0+}^{3-\alpha}u(0)=0$. Thus, it is unnecessary to include $\mathcal{I}_{0+}^{3-\alpha}u(0)=0$ in the definition of $\operatorname{dom}L$ as it is done in [11]. The condition $\mathcal{I}_{0+}^{3-\alpha}u(0)=0$, in general, does not rule out a solution that is singular at $t=0$, for example, if $\alpha=5/2$, $u(t)=t^{-1/4}$, then $\mathcal{I}_{0+}^{3-\alpha}u(0)=0$. So, if only non-singular solutions are under consideration and the domain of L is a subset of $C^{\alpha-1}$, it need not be restricted by inclusion of $\mathcal{I}_{0+}^{3-\alpha}u(0)=0$, which already is a direct consequence of $u\in C^{\alpha-1}$. An alternative to $C^{\alpha-1}$ is the space X endowed with the same norm as in [11]. Then the condition $u(0)=0$ must be included in our definition of $\operatorname{dom}L$.

We assume that the following condition holds:

(H_0) $B_i:X\to\mathbb{R}$, $i=1,2$ are linear bounded functionals with the respective norms $\|B_i\|$, $i=1,2$, satisfying $B_1(t^{\alpha-1})B_2(t^{\alpha-2})=B_1(t^{\alpha-2})B_2(t^{\alpha-1})$, where $B_1^2(t^{\alpha-1})+B_1^2(t^{\alpha-2})\neq 0$. For convenience, we introduce the constants $\beta,a,b\in\mathbb{R}$ defined by the relations $a=B_1(t^{\alpha-2})$, $b=B_1(t^{\alpha-1})$, $B_2(t^{\alpha-2})=\beta a$, $B_2(t^{\alpha-1})=\beta b$.

Remark 2. By $B_i(u)=0$, $i=1,2$, we understand that there exist Riemann-Stieltjes measures $\xi_{ij}(t)$, $i=1,2$, $j=1,2,3$, such that

$$B_i(u)=\int_0^1 u(t)\,d\xi_{i1}(t)+\int_0^1 D_{0^+}^{\alpha-2}u(t)\,d\xi_{i2}(t)+\int_0^1 D_{0^+}^{\alpha-1}u(t)\,d\xi_{i3}(t)=0,\quad i=1,2.$$

However, we do not rely on the knowledge of these measures. Since $\Phi_1,\Phi_2:C[0,1]\to\mathbb{R}$ are continuous linear functionals, the functional conditions in (2) allow similar representations as

$$\Phi_1(D_{0^+}^{\alpha-1}u)=\int_0^1 D_{0^+}^{\alpha-1}u(t)\,d\eta_1(t)=0,$$

$$\Phi_2(D_{0^+}^{\alpha-2}u)=\int_0^1 D_{0^+}^{\alpha-1}u(t)\,d\eta_2(t)=0,$$

where η_1 and η_2 are Riemann-Stieltjes measures. Hence the conditions (2) are not quite as general as (5).

If $B_1(t^{\alpha-1})=\Gamma(\alpha)\Phi_1(1)$, $B_1(t^{\alpha-2})=0$, $B_2(t^{\alpha-1})=\Gamma(\alpha)\Phi_2(t)$, $B_1(t^{\alpha-2})=\Gamma(\alpha-1)\Phi_2(1)$, then the problems (4), (5) and (1), (2) admit the same resonance conditions. It is clear that the functional problems (1), (2) and (4), (5) are closely related. In that respect, whenever we claim an improvement of a result of [11], we merely refer to applicability of our method to that in the framework of (1), (2). To be concrete, we will discuss the use of (H_4) and (H_6) in our work in comparison with (H_4) and (H_6) of [11]. Moreover, none of our existence criteria depend on the artificial conditions adopted in [11]:

(a) $\Phi_1(t)\neq 0$ (Theorem 3.2 and 3.3);
(b) $\Phi_2(t^2)\neq 0$ (Theorem 3.4);
(c) $2\Phi_1(t)\Phi_2(t^3)-3\Phi_1(t^2)\Phi_2(t^2)\neq 0$ (Theorem 3.5).

Indeed, L is a Fredholm operator and the construction of Q should not rely on the above. This is shown in Remark 3. As we have already mentioned, the main objective is to study the case (3) and the additional improvements and extensions may be found in Remarks 1–4.

In this paper we are not concerned with an analogue of (A_5) and uniqueness results.

Define operators $L:\operatorname{dom}L\subset X\to Y$, N: $X\to Y$ by

$$Lu(t)=D_{0^+}^{\alpha}u(t),\quad Nu(t)=f\big(t,u(t),D_{0^+}^{\alpha-2}u(t),D_{0^+}^{\alpha-1}u(t)\big),$$

where $\operatorname{dom}L=\{u\in X:D_{0+}^{\alpha}u\in Y,u(0)=0,B_i(u)=0,i=1,2\}$.

We recall now the essentials of the coincidence degree theory. Let X and Y be real normed spaces. A linear operator $L:\operatorname{dom}L\subset X\to Y$ is called a Fredholm operator if $\operatorname{Ker}L$ has a finite dimension and $\operatorname{Im}L$ is closed and has a finite co-dimension. The Fredholm index is the integer $\operatorname{Ind}L=\dim\operatorname{Ker}L-\operatorname{codim}\operatorname{Im}L$.

If P: $X\to X$ and Q: $Y\to Y$ are continuous linear projectors with

$$\operatorname{Im}P=\operatorname{Ker}L,\ \operatorname{Ker}Q=\operatorname{Im}L,\quad X=\operatorname{Ker}L\oplus\operatorname{Ker}P,\ Y=\operatorname{Im}L\oplus\operatorname{Im}Q,$$

and L: $X\to Y$ is a Fredholm operator of index zero, then the inverse of the operator

$$L|_{\operatorname{dom}L\cap\operatorname{Ker}P}:\operatorname{dom}L\cap\operatorname{Ker}P\to\operatorname{Im}L$$

exists and is denoted by $K_P:\operatorname{Im}L\to\operatorname{dom}L\cap\operatorname{Ker}P$. Furthermore, the operator N: $X\to Y$ is said to be L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to X$ is compact. The abstract equation $Lu=Nu$ is shown to be solvable in view of Theorem IV.13 [5]:

Theorem 1.3. *Let* $\Omega\subset X$ *be open and bounded,* L *be a Fredholm mapping of index zero and* N *be* L*-compact on* $\overline{\Omega}$*. Assume that the following conditions are satisfied:*

(i) $Lu\neq\lambda Nu$ *for every* $(u,\lambda)\in((\operatorname{dom}L\setminus\operatorname{Ker}L)\cap\partial\Omega)\times(0,1)$*;*
(ii) $Nu\notin\operatorname{Im}L$ *for every* $u\in\operatorname{Ker}L\cap\partial\Omega$*;*
(iii) $\deg(QN|_{\operatorname{Ker}L\cap\partial\Omega},\Omega\cap\operatorname{Ker}L,0)\neq 0$, *with* Q: $Y\to Y$ *a continuous projector such that* $\operatorname{Ker}Q=\operatorname{Im}L$.