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On the instability of two entropic dynamical models☆

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A R T I C L E   I N F O

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In this paper we study two entropic dynamical models from the viewpoint of information geometry. We study the geometry structures of the associated statistical manifolds. In order to analyse the character of the instability of the systems, we obtain their geodesics and compute their Jacobi vector fields. The results of this work improve and extend a recent advance in this topics studied in Peng et al.[13].

1. Introduction

The evolution of some systems could be predicted with certitude, however in some cases, by the complexity of the system, lack of information, etc, the predictions of final states can be done at the best only by assigning probabilities. Examples of these system could be found in biology, ecology, chemistry, physics, and economics. Some authors believe that quantum mechanics might be derived by the laws of probability inference, as well as happens with thermodynamic (see for instance [5] and [6]). Entropic Dynamics (see [7]) provided a tool that could be useful in the study of the dynamics of certain complex systems. Roughly, given a system, the Entropic Dynamic make use of maximum relative entropy principle in order to determine a statistical manifold that model it. This statistical manifold represent the total macro-states of the system (i.e., probability distributions). To obtain this manifold, firstly we have to determine the micro-states and the constraint of the system. For instance, if we want to study the dynamics of k particles in a l-dimensional Euclidean space, the micro-states could be the lk-random variables $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^{lk}$ with $x_i = x_i^1, \dots, x_i^l$ and distributions p_i^j that represent the position of the particles. The constraints could be the expected values or the variances of p_i^j, or some extra knowledge, for instance, if these distributions are correlated or not. These constraints are the only testable information that we can get from the system. In order to get the family of distributions that better fit to the system we maximize the relative entropy functional (see [6]) given a prior probability density (the uniform distribution). In the case that the constrains are the expected valued u_i^j and the variance v_i^j of p_i^j and assuming that x_i^j are independent distributed random variables, then we will get a statistical manifold S of dimension $2lk$ parametrized by a function ϕ over some open set of $\mathbb{R}^{2lk}$

$$\left((u_1^1, v_1^1), \dots, (u_k^l, v_k^l)\right) \longrightarrow \phi\left((u_1^1, v_1^1), \dots, (u_k^l, v_k^l)\right)$$
$$= (p_1^1, \dots, p_1^l, \dots, p_k^1, \dots, p_k^l) \in S.$$

We are going to consider the geometry of the manifold S induced by the Fisher information metric g (see Section 2 for the definition). The evolution of the system can be seen as a continuous path in S. The entropic dynamics principle claims that the system evolves followings the geodesics of the Riemannian manifold (S, g). Therefore, the curvature of (S, g) encoded some information on the dynamic of the system. So, the task is to study the geometry of (S, g) from the Information Geometry viewpoint (see [1] and [2]) in order to understand the features of the system under consideration. There are several references related with the study of entropic dynamical models from the viewpoint of information geometry, see for instance [4,5,10] among others.

Nevertheless, there does not exist a general standard procedure to set up the appropriated constraints for a given system. Most of the time this must be done by intuition or by some experimental data. So, it seen important to understand the geometry of some statistical models. In the present article we study some statistical manifolds that appear in several fields, such as physics, biology, social sciences, economics, see for instance [8,9,13–17], among others.

The aim of present article is to extend and study two entropic dynamical models introduced by Peng, Sun, Sun, and Yi in [12].

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In [12], the authors studied the character of the instability of two entropic dynamical models:

- M_1: with a statistical manifold induced by a family of a joint Gamma and Exponential distributions
- M_2: with a statistical manifold induced by a family of a joint Gamma and Gaussian distributions.

From the study of the geometry of both models, they found out that M_1 have first order linear divergent instability and M_2 have exponential instability.

The first model that we consider is given by the statistical manifold induced by the $k-$joint one parametric exponential family (it model a system of uncorrelated k particles). Second, we study a system of two correlated particles modelled by the statistical manifold of the multivariate Gaussian probability family. Finally, we discuss how these models can be combined in order to generalize the obtained results to a large class of models.

The paper is organized as follows. In Section 2, we introduce a k-dimensional statistical manifold induced by densities of a one parameter exponential family and we study its geometrical structure. We analyse the character of the stability of this model when $k=4$. In Section 3, we study the geometric structure and the stability of a Gaussian statistical manifolds with correlations. Conclusions and some extensions are presented in Section 4.

2. Geometric structure and stability of $k-$dimensional statistical manifold

We refer the reader to [1] and [11] for definitions and standard results concerning to the geometry of statistical manifolds.

We consider a system of k particles in a one dimensional space named $\mathbf{x}=(x_1,\dots,x_k)$. We assume that all information relevant to the dynamical model comes from the probability distribution which in this case is the joint distribution of k independent one parameter exponential family. More precisely, we consider the following joint density function

$$p(\mathbf{x},\boldsymbol{\theta}) = h(\mathbf{x})\exp\left(\sum_{s=1}^{k}(\eta_s(\theta_s)T_s(x_s)-\gamma_s(\theta_s))\right)$$

with $\boldsymbol{\theta}=(\theta_1,\dots,\theta_k)$, $\mathbf{x}=(x_1,\dots,x_k)$, T_s is a continuous function and η_s and γ_s are twice-differentiable functions for $s=1,\dots,k$.

$$\Gamma^i_{ii} = \frac{\gamma_i'''(\theta_i)(\eta_i'(\theta_i))^2-\gamma_i'(\theta_i)\eta_i'''(\theta_i)\eta_i'(\theta_i)-\eta_i''(\theta_i)(\gamma_i''(\theta_i)\eta_i'(\theta_i)-\gamma_i'(\theta_i)\eta_i''(\theta_i))}{2\eta_i'(\theta_i)(\gamma_i''(\theta_i)\eta_i'(\theta_i)-\gamma_i'(\theta_i)\eta_i''(\theta_i))}$$
$$=\frac{1}{2}\left(\frac{\gamma_i'''(\theta_i)\eta_i'(\theta_i)-A_i'(\theta_i)\eta_i'''(\theta_i)}{A_i''(\theta_i)\eta_i'(\theta_i)-\gamma_i'(\theta_i)\eta_i''(\theta_i)}-\frac{\eta_i''(\theta_i)}{\eta_i'(\theta_i)}\right). \tag{2}$$

Therefore, we can define the associated statistical manifold as follows

$$M_k := \left\{p(\mathbf{x},\boldsymbol{\theta}) = h(\mathbf{x})\exp\left(\sum_{s=1}^{k}(\eta_s(\theta_s)T_s(x_s)-\gamma_s(\theta_s))\right)\quad \theta_s\in\mathbb{R}\text{ for } s=1,\dots,k\right\}.$$

We are going to consider M_k endowed with the Fisher-information matrix. This metric is proportional to the amount of information that the distribution function contains about the parameter. Recall that the local expression of the Fisher-information metric with respect to the coordinate system $\boldsymbol{\theta}$ is:

$$g_{ij}(\boldsymbol{\theta}) = E\big(\partial_i l(\boldsymbol{\theta})\partial_j l(\boldsymbol{\theta})\big)$$

where $\partial_i l(\theta) = \frac{\partial}{\partial\theta_i}\log p(\mathbf{x},\boldsymbol{\theta})$. It is easy to see that the Fisher-information metric on M_k can be computed as

$$g_{ij}(\boldsymbol{\theta}) = E\big((\eta_i'(\theta_i)T_i(x_i)-\gamma_i'(\theta_i))(\eta_j'(\theta_j)T_j(x_j)-\gamma_j'(\theta_j))\big).$$

Since the variables x_s $(s=1,\dots,k)$ have density function belonging to one parameter exponential family, the expected value and the variance of T_s can be computed easily in terms of η_s and γ_s. Indeed,

$$E(T_s)=\frac{\gamma_s'(\theta_s)}{\eta_s'(\theta_s)}\qquad Var(T_s)=\frac{\gamma_s''(\theta_s)\eta_s'(\theta_s)-\gamma_s'(\theta_s)\eta_s''(\theta_s)}{(\eta_s'(\theta_s))^3}.$$

From the independence of the variables x_s we have

$$g_{ij}(\boldsymbol{\theta}) = \delta_{ij}(\eta_i'(\theta_i))^2\,Var(T_i) = \delta_{ij}\frac{\gamma_i''(\theta_i)\eta_i'(\theta_i)-\gamma_i'(\theta_i)\eta_i''(\theta_i)}{\eta_i'(\theta_i)},$$

where δ_{ij} is the Kronecker's delta. Note that we have assumed uncoupled constraints between the micro-variables. This assumptions leads to a metric tensor with trivial off diagonal elements.

The inverse matrix of g is

$$g^{-1}=[g^{ij}] = diag\left(\frac{\eta_1'(\theta_1)}{\gamma_1''(\theta_1)\eta_1'(\theta_1)-\gamma_1'(\theta_1)\eta_1''(\theta_1)},\dots,\frac{\eta_k'(\theta_k)}{\gamma_k''(\theta_k)\eta_k'(\theta_k)-\gamma_k'(\theta_k)\eta_k''(\theta_k)}\right).$$

The length element is given by

$$ds^2 = g_{ij}d\theta_i\theta_j = \sum_i \frac{\gamma_i''(\theta_i)\eta_i'(\theta_i)-\gamma_i'(\theta_i)\eta_i''(\theta_i)}{\eta_i'(\theta_i)}\,d\theta_i^2,$$

and the volume element is

$$dV_g = \sqrt{g}\,d\theta_1\wedge\cdots\wedge d\theta_k = \left(\prod_i\frac{\gamma_i''(\theta_i)\eta_i'(\theta_i)-\gamma_i'(\theta_i)\eta_i''(\theta_i)}{\eta_i'(\theta_i)}\right)^{1/2} d\theta_1\wedge\cdots\wedge d\theta_k. \tag{1}$$

where $\sqrt{g}=\sqrt{\det(g_{ij})}$.

Recall that the Christoffel symbols Γ^l_{ij} is defined by $\Gamma^l_{ij}=\Gamma_{ijs}g^{sl}$ $(i,j,l,s=1,2,\dots,k)$ where

$$\Gamma_{ijs}=\frac{1}{2}(\partial_i g_{js}+\partial_j g_{si}-\partial_s g_{ij}),\qquad i,j,s=1,\dots,k.$$

For this model the Christoffel symbols that are not zero are:

The Ricci curvature R_{is} is defined by $R_{is}=R_{ijsl}g^{jl}$ $i,j,s,l=1,\dots,k$ where

$$R_{ijsl}=(\partial_j\Gamma^u_{is}-\partial_i\Gamma^u_{js})g_{ul}+(\Gamma_{jtl}\Gamma^t_{is}-\Gamma_{itl}\Gamma^t_{js}).$$

Therefore, it is easy to see that the curvature tensor components are all zero and the scalar curvature $S_g=0$.

Recall that the geodesic equations are given by the following non linear system of second order ordinary differential equations:

$$\frac{\partial^2\theta_l}{\partial\tau^2}+\Gamma^l_{ij}\frac{\partial\theta_i}{\partial\tau}\frac{\partial\theta_j}{\partial\tau}=0\qquad\text{for } i,j,l=1\dots,k. \tag{3}$$

From (2) we obtain that the geodesics are determined by the following k differential equations:

$$\frac{\partial^2\theta_i}{\partial^2\tau}+\Gamma^i_{ii}\left(\frac{\partial\theta_i}{\partial\tau}\right)^2=0\qquad\text{for}\qquad i=1,\dots,k. \tag{4}$$