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VALIDITY of the multifractal formalism under the box condition

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ABSTRACT

In this paper, we intend to generalise the work of Barral et al. (2003) [1], which provides a bridge between the *c*-adic boxes and the grid-free approaches to the multifractal analysis of measures. More precisely, we consider some sort of an irregular grid. We apply our results to a Bernoulli measure.

1. Introduction and preliminaries

The multifractal analysis is a natural framework to describe geometrically the heterogeneity in the distribution at small scales of positive and finite compactly supported Borel measures on a metric spaces X. Specifically, for such a measure μ, one considers the level sets of the pointwise Hölder exponent of μ, this heterogeneity can be classified by considering the iso-Hölder sets

$$X_\mu(\alpha) = \left\{x \in \mathrm{Supp}\mu / \lim_{r \longrightarrow 0} \frac{\log(\mu(B(x,r))}{\log r} = \alpha\right\}$$

Then, the singularity spectrum of μ is the mapping

$$\alpha \geq 0 \longmapsto \dim(X_\mu(\alpha))$$

where dim stands for the Hausdorff dimension. We say that the multifractal formalism is valid if $\dim(X_\mu(\alpha))$ is equal to the Legendre transform at α of a scale function associated to μ. In [3], in order to prove the validity of the centred multifractal formalism of Olsen [7], the authors imposed a condition on the exterior generalised Hausdorff measure $\mathcal{H}_\mu^{q,B_\mu(q)}$, more precisely, they proved that

$$\text{if } \mathcal{H}_\mu^{q,B_\mu(q)}(\mathrm{Supp}\mu) > 0 \text{ then } \dim(X_\mu(-B'_\mu(q))) = B^*_\mu(-B'_\mu(q))$$

where B^*_μ is the Legendre transform of B_μ. Unfortunately, this hypothesis is often difficult to verify because the measure is in general constructed generally over boxes. In [1], the authors replaced the hypothesis $\mathcal{H}_\mu^{q,B_\mu(q)}(\mathrm{Supp}\mu) > 0$ by conditions depending only on the measure μ and the $c-$adic grid. More precisely, they showed that under the neighbouring box condition (NBC) hypothesis and if there exists a Frostman measure associated to μ for the c-adic boxes, the centred multifractal formalism is valid. Notice that the authors have been applied these results to quasi-Bernoulli and Mandelbrot measures ([1]).

In this work, we propose to generalise results of ([1]) to an irregular grid. Then we will apply our main theorem to a Bernoulli measure. Let's give a brief description of the organisation of the paper. For the convenience of the reader we recall the Olsen's multifractal formalism ([7]) in Section 2. Section 3 deals with box formalism. In Section 4 we state our main results and their proofs. In Section 5, we prove the validity of the multifractal formalism for a Bernoulli measure constructed on an irregular grid.

2. A centred multifractal formalism

Definition 2.1. A metric space (X, d) is said to have the Besicovich covering property if there exists a positive integer ξ such that, given any collection $\{B(x_j, r_j)\}_j$ of balls, one can extract from it, ξ packings $P_1, P_2, P_3, \ldots, P_\xi$ which altogether form a cover of the set $\{x_j\}_j$.

Let (X, d) be a metric space having the Besicovich covering and μ a positive atomless Borel measure on X. According to Olsen ([7]), we define several measures and premeasures

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indexed by a couple (q, t) of real numbers. If E is a subset of X and δ is a positive real number, we set

$$\overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E) = \sup \sum_j \mu(B(x_j, r_j))^q (2r_j)^t$$

with the convention, $0^q = 0$ for all $q \in \mathbb{R}$, which is valid throughout this paper.

Notice that this supremum is being taken over the collections $\{B(x_j, r_j)\}_j$ of centred δ-packing composed of mutually closed disjoint balls of E whose centers x_j belong to E and whose diameter $2r_j$ are less than δ. We consider the limit

$$\overline{\mathcal{P}}^{q,t}_{\mu}(E) = \lim_{\delta \longrightarrow 0} \overline{\mathcal{P}}^{q,t}_{\mu,\delta}(E)$$

The function $\overline{\mathcal{P}}^{q,t}_{\mu}$ is called packing pre-measure. It is increasing and it lacks σ-subadditivity to be a Caratheodory outer measure. That is why one considers the following quantity

$$\mathcal{P}^{q,t}_{\mu}(E) = \inf_{E \subset \cup_j E_j} \sum_j \overline{\mathcal{P}}^{q,t}_{\mu}(E_j)$$

which, as a function of E, is an outer measure. This is the same process as for defining packing measures, which were introduced in ([9]). In a similar way, one defines the generalised Hausdorff measure with respect to q and t.

$$\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) = \inf \sum_j \mu(B(x_j, r_j))^q (2r_j)^t.$$

This infimum is being taken over all δ- centred covering $\{B(x_j, r_j)\}_j$ of E by balls such that $E \subset \cup_j B(x_i, r_j)$, whose centers x_j belong to E and whose diameter $2r_j$ are less than δ and consider the limit

$$\overline{\mathcal{H}}^{q,t}_{\mu}(E) = \lim_{\delta \longrightarrow 0} \overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E)$$

The function $\overline{\mathcal{H}}^{q,t}_{\mu}$ is σ-subadditive but not increasing. In order to deal with an outer measure, one defines

$$\mathcal{H}^{q,t}_{\mu}(E) = \sup_{F \subset E} \overline{\mathcal{H}}^{q,t}_{\mu}(F)$$

These last measures are the multifractal counterparts of the centred Hausdorff measures introduced in [8]. For a fixed q, if, for some t, one has $\overline{\mathcal{P}}^{q,t}_{\mu}(\mathrm{Supp}\mu) < +\infty$, then, for all $t' > t$, one has $\overline{\mathcal{P}}^{q,t'}_{\mu}(\mathrm{Supp}\mu) = 0$. Where $\mathrm{Supp}\mu$ stands for the topological support of μ. Therefore, there exists a unique $\Lambda_\mu(q) \in \overline{\mathbb{R}}$ such that

$$\Lambda_\mu(q) = \sup\{t \in \mathbb{R} / \overline{\mathcal{P}}^{q,t}_{\mu}(\mathrm{Supp}\mu) = +\infty\}$$

In a similar way, there exist two functions B_μ and b_μ respectively associated with $\mathcal{P}^{q,t}_{\mu}$ and $\mathcal{H}^{q,t}_{\mu}$ such that

$$B_\mu(q) = \sup\{t \in \mathbb{R} / \mathcal{P}^{q,t}_{\mu}(\mathrm{Supp}\mu) = +\infty\}$$

and

$$b_\mu(q) = \sup\{t \in \mathbb{R} / \mathcal{H}^{q,t}_{\mu}(\mathrm{Supp}\mu) = +\infty\}$$

Record that all these three functions are non increasing and Λ_μ and B_μ are convex. It is clear that $B_\mu \leq \Lambda_\mu$.

Remark that if moreover the metric space (X, d) has the Besicovich covering property defined previously, one has $b_\mu \leq B_\mu$, and we conclude that

$$b_\mu \leq B_\mu \leq \Lambda_\mu \tag{2.1}$$

If α and β are two real numbers such that $\alpha \leq \beta$, one considers the following sets

$$X_\mu(\alpha, \beta) = \left\{ x \in \mathrm{Supp}\mu / \alpha \leq \liminf_{r \longrightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \limsup_{r \longrightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \beta \right\}$$

Instead of $X_\mu(\alpha, \alpha)$, we shall simply write $X_\mu(\alpha)$.

If the derivative of B_μ exists at point q, it is known [4,7] that the following inequalities hold, if $X_\mu(-B'_\mu(q)) \neq \emptyset$

$$\begin{aligned} \dim X_\mu(-B'_\mu(q)) &\leq b^*_\mu(-B'_\mu(q)) \\ Dim X_\mu(-B'_\mu(q)) &\leq B^*_\mu(-B'_\mu(q) \end{aligned} \tag{2.2}$$

where dim and Dim stand for the Hausdorff and packing dimensions ([8], [9]), and the star as an exponent denotes the Legendre transform, i.e.

$$f^*(\alpha) = \inf_{q \in \mathbb{R}} (\alpha q + f(q)), \quad x \in \mathbb{R}$$

Definition 2.2. If $B'_\mu(q)$ exists and if all the quantities in (2.2) are equal, one says that the measure μ obeys the multifractal formalism at point q.

In [3], Ben Nasr et al., have established the following result.

Theorem (3). *Suppose that (X, d) has the Besicovich covering property, then if $\alpha = -B'_\mu(q)$ exists and $\mathcal{H}^{q,B_\mu(q)}_{\mu}(\mathrm{Supp}\mu) > 0$, one has*

$$\dim X_\mu(\alpha) = Dim X_\mu(\alpha) = B^*_\mu(\alpha) \quad \text{and} \quad b_\mu(q) = B_\mu(q)$$

3. Box analysis

Let $\mathcal{F} = \cup_{n \geq 1} \mathcal{F}_n$ such that $\{\mathcal{F}_n\}_{n \geq 1}$ stands for a sequence of nested finite partition of $[0, 1[$, by semi-open intervals. If x belongs to $[0, 1[$, $I_n(x)$ stands for the interval in $\mathcal{F}_n$ which contains x. If I is the Borel set in $[0, 1[$, $|I|$ denotes its Lebesgue measure.

We suppose that:

H_1) $\mathcal{F}_{n+1}$ is a refinement of $\mathcal{F}_n$ and for all $x \in [0, 1[$; $\lim_{n \to +\infty} |I_n(x)| = 0$.

H_2) There exist $s \in \mathbb{N} \backslash \{0\}$ and $L > 0$ such that for every $x \in]0, 1[$ and for all $0 < r \leq 1$, there exist k contiguous intervals of $\mathcal{F}$, $J_1, J_2, \ldots, J_k$ $(k \leq s)$, of the same generation, satisfying

1. $\forall\ 1 \leq i \leq k, \quad \frac{1}{L} r \leq |J_i| \leq Lr.$
2. $B(x, 2r) \cap [0, 1[\subset J_1 \cup J_2 \cdots \cup J_k.$
3. $\forall\ y \in B(x, r) \cap]0, 1[,$

$$\begin{cases} \exists\ 1 \leq i \leq k;\ J_i \subset B(y, r) \text{ with } y \in \mathring{J}_i \\ \text{or} \\ \exists\ 1 \leq i_1, i_2 \leq k;\ J_{i_1} \text{ and } J_{i_2} \text{ contiguous and} \\ y \in J_{i_1} \cup J_{i_2} \subset B(y, r) \end{cases}$$

where $\mathring{J}_i$ stands for the interior of J_i.