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Achievement sets on the plane—Perturbations of geometric and multigeometric series

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ABSTRACT

By $A(x_n) = \{\sum_{n=1}^{\infty} \varepsilon_n x_n : \varepsilon_n = 0, 1\}$ we denote the achievement set of the absolutely convergent series $\sum_{n=1}^{\infty} x_n$. We study the relation between the achievement set of the series on the plane and the achievement sets of its projection into two coordinates. We mainly focus on the series $\sum_{n=1}^{\infty}(x_n, y_n)$ where (x_n) is a geometric series and $y_n = x_{\sigma(n)}$ for some permutation $\sigma \in S_\infty$.

If (x_n) is a multigeometric sequence, then $A(x_n, x_{\sigma(n)})$ can be one of at least seven types of sets, which are strongly related to three types of attainable achievement sets on the real line. We conjecture that if (x_n) multigeometric, then $A(x_n, x_{\sigma(n)})$ can be one of eight types – none of them homeomorphic to the other one.

We prove a general fact on the Hausdorff dimension of the achievement set in Banach spaces. As a corollary we obtain that if $0 < q \le 1/2$, $\dim_H(A(q_n, q_{\sigma(n)})) = \dim_H(A(x_n)) = -\log 2/\log q$ for some class of regular permutations $\sigma \in S_\infty$.

1. Introduction

Suppose that $x = (x_n)_{n=1}^{\infty} \in \ell_1$ and let

$$A(x) = \left\{ \sum_{n=1}^{\infty} \varepsilon_n x_n : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}} \right\}$$

denote the set of all subsums of the series $\sum_{n=1}^{\infty} x_n$, called *the achievement set* (or *a partial sumset*) of x, *see* [7]. In 1914 Kakeya [8] initiated the study of topological properties of achievement sets presenting the following result:

Theorem 1.1 (*Kakeya*). *For any sequence $x \in \ell_1 \setminus c_{00}$*

(1) *$A(x)$ is a perfect compact set.*
(2) *If $|x_n| > \Sigma_{i>n}|x_i|$ for almost all n, then $A(x)$ is homeomorphic to the ternary Cantor set.*
(3) *If $|x_n| \le \Sigma_{i>n}|x_i|$ for almost all n, then $A(x)$ is a finite union of closed intervals. In the case of non-increasing sequence x, the last inequality is also necessary for $A(x)$ to be a finite union of intervals.*

Kakeya conjecture was that $A(x)$ is either nowhere dense or a finite union of intervals. It was disproved by Weinstein and Shapiro [14] and, independently, by Ferens [5]. Guthrie and Nymann in [6] gave a simple example of sequence, namely $x = \left(\frac{5+(-1)^n}{4^n}\right)_{n=1}^{\infty}$, such that its achievement set $T = A(x)$ contains an interval but it is not a finite union of intervals. In the same paper the authors formulated the following trichotomy for achievement sets, finally proved in Neymann and Saenz [12]:

Theorem 1.2. *For any sequence $x \in \ell_1 \setminus c_{00}$, $A(x)$ is one of the following sets*:

(1) *a finite union of closed intervals;*
(2) *homeomorphic to the ternary Cantor set;*
(3) *homeomorphic to the set T.*

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The set T is homeomorphic to $C \cup \bigcup_{n=1}^{\infty} S_{2n-1}$, where S_n denotes the union of the 2^{n-1} open middle thirds which are removed from [0, 1] at the nth step in the construction of the ternary Cantor set C. Such sets are called Cantorvals. Formally, a *Cantorval* (more precisely, an $\mathcal{M}$-Cantorval, see [9]) is a non-empty compact subset S of the real line, such that S is the closure of its interior, and both endpoints of any infinite component are accumulation points of one-point components of S. A non-empty subset C of the real plane will be called a *Cantor set* if it is compact, zero-dimensional and has no isolated points.

Note that Theorem 1.2*says that ℓ_1 can be divided into four sets: c_{00} and the sets with properties prescribed in (1), (2) and (3). Some algebraic and topological properties of these sets have been recently considered in [2].*

The sequence of the form $(k_1, k_2, \ldots, k_m, k_1q, \ldots, k_mq, k_1q^2, \ldots)$ is called multigeometric sequence (see [3]) and it is denoted by $(k_1, k_2, \ldots, k_m; q)$. Note that Guthrie–Nymann sequence $\left(\frac{5+(-1)^n}{4^n}\right)_{n=1}^{\infty}$ is a multigeometric series of the form (3/4, 6/4; 1/4). If $k_1 = \ldots = k_m$, then by Kakeya Theorem $\mathrm{A}(k_1, k_2, \ldots, k_m; q)$ is either a Cantor set or an interval. As in Banakh et al. [1] we denote by Σ the set

$$\left\{\sum_{n=1}^{m} k_n\varepsilon_n : (\varepsilon_n)_{n=1}^{m} \in \{0,1\}^m\right\}.$$

Let us write Σ as $\{\tau_1 < \ldots < \tau_s\}$. Then the one-dimensional achievement set $\mathrm{A}(x)$ depends only on Σ and the ratio q. We consider the following numbers connected with Σ: $\mathrm{diam}(\Sigma) = \tau_s - \tau_1$, $\Delta(\Sigma) = \max_{i<s}(\tau_{i+1} - \tau_i)$ and $I(\Sigma) = \Delta(\Sigma)/(\Delta(\Sigma) + \mathrm{diam}(\Sigma))$. Moreover, we have $|\Sigma| = s$. It was proved in Banakh et al. [1] that

(1) A is an interval if and only if $q \geq I(\Sigma)$.
(2) A is not a finite union of intervals if $q < I(\Sigma)$ and $\Delta(\Sigma) \in \{\tau_2 - \tau_1, \tau_s - \tau_{s-1}\}$.
(3) A is a Cantor set of zero Lebesgue measure if $q < 1/s$.

For a metric space (X, ρ) by $\mathcal{K}(X)$ we denote the hyperspace of all non-empty compact subsets of X. There is a natural metric on $\mathcal{K}(X)$, namely the Hausdorff distance given by

$$\rho_H(K, L) = \inf\{\delta > 0 : L \subset B(K, \delta) \text{ and } K \subset B(L, \delta)\}$$

where $K, L \in \mathcal{K}(X)$ and $B(K, \delta) = \bigcup_{x \in K} B(x, \delta)$ is a δ-neighborhood of K. The iterated function system fractal (in short IFS fractal) generated by the system of affine contractions $\{f_1, \ldots, f_n\}$ is the unique fixed point of the self-map $K \mapsto \bigcup_{i=1}^{n} f_i(K)$. For a positive real number s and $\delta > 0$ define $\mathcal{H}_\delta^s(F) = \inf\{\sum_{n=1}^{\infty}(\mathrm{diam} A_n)^s : A_1, A_2, \ldots$ is a δ-cover of $F\}$ where δ-cover of F is a sequence A_1, A_2, … of sets such that $F \subset \bigcup_{n=1}^{\infty} A_n$ and $\mathrm{diam}(A_n) \leq \delta$. The s-dimensional Hausdorff outer measure is defined as $\mathcal{H}^s(F) = \lim_{\delta\to 0}\mathcal{H}_\delta^s(F) = \sup_{\delta>0}\mathcal{H}_\delta^s(F)$. It is well-known that for a given Borel set F and for $0 < s < t$, if $\mathcal{H}^s(F) < \infty$, then $\mathcal{H}^t(F) = 0$, and if $\mathcal{H}^t(F) > 0$, then $\mathcal{H}^s(F) = \infty$. The Hausdorff dimension $\dim_H(F)$ of a Borel set F is a critical value $s_0 \in [0, \infty]$, such that $\mathcal{H}^s(F) = \infty$ for all $s < s_0$ and $\mathcal{H}^s(F) = 0$ for all $s > s_0$.

Nitecki at the end of his nice survey paper [13] on subsum sets wrote: "One might also be tempted to ask about the analogous question for null sequences in the complex plane (or more generally points in $\mathbb{R}^n$). In this context (...) the analysis of translations will be made more complicated by the need to consider directions as well as distances. Who knows where that might lead?" Following this suggestion we start investigation of multidimensional achievement sets - its topological and geometric properties.

The aim of our paper is to study the properties of the achievement sets on the plane. Let $(x_n, y_n) \in \ell_1 \times \ell_1$. By

$$\mathrm{A}(x_n, y_n) := \left\{\sum_{n=1}^{\infty} \varepsilon_n(x_n, y_n) : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}\right\}$$

we denote the achievement set of the series $\sum_{n=1}^{\infty}(x_n, y_n)$. The main and the most general question we are interested in, is the following:

Problem 1.3. *Let $x_n, y_n \in \ell_1$ be such that $A(x_n) = C_1$ and $A(y_n) = C_2$. What can be said about $A(x_n, y_n)$?*

Achievement sets of series in $\mathbb{R}^n$ were studied by Morán in [10] and [11]. In Morán [10] a series $\sum_{i=1}^{\infty} x_i$ is called *fractal series* if $\mathrm{A}(x_i)$ has cardinality continuum (equivalently $(x_i) \notin c_{00}$) and it has n-dimensional Lebesgue measure zero. The author has given some sufficient conditions for series $\sum_{i=1}^{\infty} x_i$ being a fractal series. Each of them implies that $\sum_{i=1}^{\infty} x_i$ is quickly convergent, which means $\|x_i\| > \Sigma_{k>i}\|x_k\|$ for almost every i, which is a Kakeya type condition. Morán has estimated, and in some cases precisely calculated, the Hausdorff dimension of the achievement sets.

It is easy to observe that, as in one-dimensional case, the achievement set on the plane is a compact perfect set (or finite set if elements of underlying series are eventually zero). Moreover, the set $\mathrm{A}(x_n, y_n)$ is contained in $C_1 \times C_2$ – the Cartesian product of achievement sets of (x_n) and (y_n), and $\mathrm{A}(x_n, y_n)$ is symmetric with respect to the middle point of $C_1 \times C_2$. Thus if $\mathrm{A}(x_n)$ and $\mathrm{A}(y_n)$ are Cantor sets, so is $\mathrm{A}(x_n, y_n)$.

If $\mathrm{A}(x_n) = C$, then $\mathrm{A}(x_n, x_n) = \sqrt{2}\mathrm{R}_{\frac{\pi}{4}}(C)$ where $\mathrm{R}_{\frac{\pi}{4}}$ is the anticlockwise rotation around the origin at an angle of $\frac{\pi}{4}$. On the other hand if one add zeros to the series x_n, then the one-dimensional achievement set remains unchanged. In particular

$$\begin{aligned}\mathrm{A}(x_1, 0, x_2, 0, x_3, 0, \ldots) &= \mathrm{A}(0, x_1, 0, x_2, 0, x_3, \ldots)\\ &= \mathrm{A}(x_1, x_2, x_3, \ldots) = C\end{aligned}$$

and

$$\mathrm{A}((x_1, 0), (0, x_1), (x_2, 0), (0, x_2), (x_3, 0), (0, x_3), \ldots) = C \times C.$$

This simple observation shows that to get something interesting we need to make some restrictions on the sequence (x_n, y_n). We will deal with the following more specific question.

Problem 1.4. *Let $x_n > 0$ for every $n \in \mathbb{N}$. Assume that $A(x_n) = C$. What can be said about $A(x_n, x_{\sigma(n)})$ where $\sigma \in S_\infty$?*

In this paper we will consider even more specific situation. Namely we will consider the case when the series $\sum_{n=1}^{\infty} x_n$ is a geometric or multigeometric series and we will restrict our attention to permutations $\sigma \in S_\infty$ which are quite regular. For $q \in (0, 1)$ the series $\sum_{n=1}^{\infty}(q^n, q^{\sigma(n)})$ will