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Short communication

A derivation of the Maxwell–Cattaneo equation from the free energy and dissipation potentials

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A B S T R A C T

A thermodynamic derivation is presented of the Maxwell–Cattaneo equation involving a material time, instead of a partial time, derivative of heat flux. The Ansatz is given by the functionals of free energy and dissipation potentials, relying on an extended state space and a representation theory of Edelen.

Recently, Christov and Jordan [1] have shown that the Maxwell–Cattaneo equation governing the propagation of second sound should involve a material time derivative of heat flux ($\dot{q} \equiv Dq/Dt$) instead of a partial time derivative ($\partial q/\partial t$). That is, supposing we look at a one-dimensional (1-D) setting, this equation should read

$$q + t_0 \frac{Dq}{Dt} = -k\theta_{,x}, \tag{1}$$

where θ is the absolute temperature, t_0 is the relaxation time, and k is the thermal conductivity.

A question arises: Can (1) be justified by thermodynamics directly? In particular, can it be derived from two functionals playing roles of potentials: the free energy ψ and the dissipation function ϕ? It turns out that this cannot be done using the thermodynamic orthogonality within the framework of thermodynamics with internal variables (TIV) [2], even when the thermodynamic state space is extended to include the heat flux or another quantity (e.g. the temperature rate). It is understood [3] that an extension of that type is needed, but, to the best of our knowledge, a derivation has not yet been published. Interestingly, while extended thermodynamics readily involves broader state spaces than other theories, the equation we typically see there (e.g. [4]) involves a partial derivative of q:

$$q + t_0 \frac{\partial q}{\partial t} = -k\theta_{,x}. \tag{2}$$

Consistent with the said extension, we assume the (specific, per unit mass) internal energy u to be a function of the strains E_{ij}, the entropy η and the heat flux q_i

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$$u = e(E_{ij}, \eta, q_i) \tag{3}$$

and the (specific, per unit mass) dissipation ϕ to be a function of the heat flux and possibly its rate:

$$\phi = \phi(q_i, \dot{q}_i). \tag{4}$$

We are focusing on a rigid conductor, so that in the above we do not need to admit other fluxes or velocities.

Now, whether we use TIV or a rational thermodynamics approach, in 3-D we obtain the Clausis–Duhem inequality in the form

$$-\frac{q_i\theta_{,i}}{\theta} - \frac{\partial\psi}{\partial q_i}\dot{q}_i \geqslant 0. \tag{5}$$

Clearly, this may be written as

$$\mathbf{Y}\cdot\mathbf{v} \geqslant 0, \tag{6}$$

where

$$\mathbf{Y} = \left(-\frac{\nabla\theta}{\theta}, -\nabla_{\mathbf{q}}\psi\right) \tag{7}$$

is a vector of dissipative thermodynamic forces, and

$$\mathbf{v} = (\mathbf{q}, \dot{\mathbf{q}}) \tag{8}$$

is a vector of conjugate thermodynamic velocities. In (7) $\nabla_{\mathbf{q}}$ stands for the gradient in the space of heat flux $\mathbf{q}$. See also in [3, pp. 73–74].

A general procedure based on the representation theory due to Edelen [5–7] allows a derivation of the most general form of the constitutive relation either for $\mathbf{v}$ as a function of $\mathbf{Y}$ or for $\mathbf{Y}$ as a function of $\mathbf{v}$, subject to (6). If we are to pursue the second alternative, the following steps are involved: Assume $\mathbf{Y}$ to be a function of $\mathbf{v}$, and determine it as

$$\mathbf{Y} = \nabla_{\mathbf{v}}\phi + \mathbf{U}, \text{ or } \quad Y_i = \frac{\partial\phi}{\partial v_i} + U_i, \tag{9}$$

where the vector $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2)$ does not contribute to the entropy production

$$\mathbf{U}\cdot\mathbf{v} = 0, \tag{10}$$

while the dissipation function is

$$\phi = \int_0^1 \mathbf{v}\cdot\mathbf{Y}(\tau\mathbf{v})d\tau \tag{11}$$

and $\mathbf{U}$ is uniquely determined, for given $\mathbf{Y}$, by

$$U_i = \int_0^1 \tau v_j\left[\frac{\partial Y_i(\tau\mathbf{v})}{\partial v_j} - \frac{\partial Y_j(\tau\mathbf{v})}{\partial v_i}\right]d\tau \quad \text{with} \quad \frac{\partial[Y_i(\tau\mathbf{v}) - U_i]}{\partial v_j} = \frac{\partial\left[Y_j(\tau\mathbf{v}) - U_j\right]}{\partial v_i}. \tag{12}$$

The symmetry relations $(12)_2$ reduce to the classical Onsager reciprocity conditions

$$\frac{\partial Y_i(\tau\mathbf{v})}{\partial v_j} = \frac{\partial Y_j(\tau\mathbf{v})}{\partial v_i} \tag{13}$$

if and only if $\mathbf{U} = \mathbf{0}$.

Focusing first on the 1-D case (with $\mathbf{v}$ becoming $(q, \dot{q})$), the simplest $\mathbf{U}$ satisfying (10) is

$$U_1 = \frac{\lambda t_0}{\theta}\dot{q}, \quad U_2 = -\frac{\lambda t_0}{\theta}q, \tag{14}$$

whereby the dissipation function is a quadratic form

$$\phi(\mathbf{v}) \equiv \phi(q, \dot{q}) = \frac{1}{2\theta}\lambda q^2 + \frac{1}{2}G\dot{q}^2. \tag{15}$$

On account of (9), we obtain

$$\begin{aligned} -\frac{\theta_{,x}}{\theta} &\equiv Y_1 = \frac{\lambda q}{\theta} + U_1 = \frac{\lambda}{\theta}q + \frac{\lambda t_0}{\theta}\dot{q}, \\ -\frac{\partial\psi}{\partial q} &\equiv Y_2 = G\dot{q} + U_2 = G\dot{q} - \frac{\lambda t_0}{\theta}q. \end{aligned} \tag{16}$$

Now, observe: