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Turbulences and strict return trajectory types of interval mappings[☆]

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ABSTRACT

In this note, we investigate the number of the strict return trajectory types with order n which are turbulent of interval mappings and show that the probability that a strict return trajectory type with order n is turbulent converges to 1 as n goes to infinity.

1. Introduction

Throughout, denote by $C^0(I)$ the set of continuous self-mappings on an interval I of the real line. Let $f \in C^0(I)$ and sequence $\mathcal{P}_n = (x_0, x_1, x_2, \ldots, x_n)$ of points in I with $x_i = f(x_{i-1})$ for every $1 \le i \le n$. $\mathcal{P}_n$ is said to be a return trajectory of f with order n if $x_1 < x_0 \le x_n$ (or $x_n \le x_0 < x_1$) (see [6]). $\mathcal{P}_n$ is said to be a positively (resp. negatively) strict return trajectory, shortly written PSRT (resp. NSRT), of f with order n if $x_i < x_0 < x_n$ for every $1 \le i \le n-1$ (resp. $x_n < x_0 < x_i$ for every $1 \le i \le n-1$). If $x_i < x_0 = x_n$ for every $1 \le i \le n-1$ (resp. $x_n = x_0 < x_i$ for every $1 \le i \le n-1$), then $\mathcal{P}_n$ is said to be a periodic orbit of f with period n.

In 1964, Sharkovskii found the following order relation in the set $\mathbf{N}$ of the natural numbers

$$\begin{aligned} &3 \succ 5 \succ 7 \succ \cdots \succ 3\cdot 2 \succ 5\cdot 2 \succ 7\cdot 2 \succ \\ &\cdots \succ 3\cdot 2^2 \succ 5\cdot 2^2 \succ 7\cdot 2^2 \succ \\ &\cdots \succ 3\cdot 2^k \succ 5\cdot 2^k \succ 7\cdot 2^k \succ \cdots \succ 2^k \succ \\ &\cdots \succ 2^4 \succ 2^3 \succ 2^2 \succ 2 \succ 1 \end{aligned}$$

and obtained the following theorem.

Theorem 1.1 (Sharkovskii's theorem, see [7]). *Let $f \in C^0(I)$. For any $m, n \in \mathbf{N}$ with $n \succ m$, if f has a periodic orbit with period n, then f has a periodic orbit with period m. Moreover, for every k, there exists a mapping $f \in C^0(I)$ that has a periodic orbit with period k but does not have any periodic orbit with period j for any $j \succ k$.*

Definition 1.2 (see [3,4]). $f \in C^0(I)$ is said to be turbulent if there exist $a, b, c, d \in I$ with $a < b \le c < d$ such that $f([a, b]) \cap f([c, d]) \supset [a, d]$.

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Remark 1.3. It follows from [3] that if $f \in C^0(I)$ is turbulent, then $\mathrm{Card}(\{x \in I : f(x) = x\}) \geq 2$, where $\mathrm{Card}(A)$ denotes the cardinality of the set A.

It is well known that turbulence plays an important role in one dimensional dynamics since the relation between turbulence and the existence of periodic orbits. In [3], Block and Coppel showed that if $f \in C^0(I)$ and f^m is turbulent for some $m \in \mathbf{N}$, then f^m has periodic orbits of all periods and f is chaotic. In [1], Blokh showed that if $f \in C^0(I)$ is turbulent, then for any $m, n \in \mathbf{N}$ with $m < n$, f has a periodic orbit P with rotation pair (m, n), where n is the period of P and $m = \mathrm{Card}(\{y \in P: f(y) < y\})$. In [2], Blokh and Misiurewicz showed that if $f \in C^0(I)$ is turbulent, then for any $m, n \in \mathbf{N}$ with $2m \leq n$, f has a periodic orbit P with over-rotation pair (m, n), where n is the period of P and $2m = \mathrm{Card}(\{y \in P : (f(y) - y)(f^2(y) - f(y)) < 0\})$.

Let $\mathcal{P}_n = (x_0, x_1, x_2, \ldots, x_n) = \{y_{n-1} < \cdots < y_1 < y_0 = x_0 < y_n = x_n\}$ (resp. $\{y_n = x_n < y_0 = x_0 < y_1 < \cdots < y_{n-1}\}$) be a PSRT (resp. NSRT) of f with order n. Define map $\pi : \{0, 1, \ldots, n-1\} \longrightarrow \{1, 2, \ldots, n\}$ by $\pi(k) = j$ if $f(y_k) = y_j$. From elementary combinatorics there are $(n-1)!$ PSRT (resp. NSRT) types with order n. We say that a SRT type is turbulent if every $f \in C^0(I)$ with a SRT of that type is turbulent.

In this note, we study the methods that calculates the number of the strict return trajectory types with order n which are turbulent and obtain the following theorem.

Theorem 1.4. *Let $P(n)$ be the number of PSRT (resp. NSRT) types with order n which are turbulent. Then* $\lim_{n\longrightarrow\infty} \frac{P(n)}{(n-1)!} = 1$.

Remark 1.5. Theorem 1.4 implies that the probability that a strict return trajectory type with order n is turbulent converges to 1 as n goes to infinity. Using a computer program to test a million of the 15! PSRT (resp. NSRT) types with order 16, 99.9% of them are turbulent.

2. The number of SRT types with order *n* which are turbulent

In this section, we study the methods that calculates the number of the strict return trajectory types with order n which are turbulent and show the main results. We first show the following lemmas.

Lemma 2.1. *Let $f \in C^0(I)$ and $\mathcal{P}_n = (x_0, x_1, x_2, \ldots, x_n)$ be a SRT of f. If there exist $x_i < x_j$ with $i, j \in \{0, 1, \ldots, n-1\}$ such that $f(x_i) = x_{i+1} < x_i < x_j < x_{j+1} = f(x_j)$, then f is turbulent.*

Proof. We may assume without loss of generality that $x_n < x_0 < x_1$, $\mathcal{P}_n \cap (x_i, x_j) = \emptyset$ and $p = \max\{x \in [x_i, x_j] : x = f(x)\}$. Set $k = \min\{r > j+1 : x_r \leq p\}$. Then $k \geq j+2$ and $x_{k-1} > p$. Write $s = \min\{j \leq r \leq k-1 : x_r \geq x_{k-1}\}$. Then $s \geq 1$ and $p < x_{s-1} < x_{k-1}$ with $f([p, x_{s-1}]) \cap f([x_{s-1}, x_{k-1}]) \supset [p, x_{k-1}]$. The proof is completed. □

From Lemma 2.1 we see that if there exist $i, j \in \{0, 1, \ldots, n-1\}$ such that $\pi(i) < i < j < \pi(j)$, then π is turbulent. On the other hand, for a PSRT type (resp. NSRT type) π with order n, if there exists $k \in \{0, 1, \ldots, n-2\}$ such that $\pi(j) < j$ for any $0 \leq j \leq k$ and $j < \pi(j)$ for any $k+1 \leq j \leq n-1$ (resp. $j < \pi(j)$ for any $0 \leq j \leq k$ and $\pi(j) < j$ for any $k+1 \leq j \leq n-1$), then π is not turbulent. In fact, we can construct a mapping $f \in C^0([1-n, 1])$ (resp. $g \in C^0([-1, n-1])$) satisfying

(i) $f(-i) = -\pi(i)$ if $i \in \{0, 1, \ldots, n-1\}$ with $\pi(i) \neq n$ and $f(-i) = 1$ if $\pi(i) = n$. $g(i) = \pi(i)$ if $i \in \{0, 1, \ldots, n-1\}$ with $\pi(i) \neq n$ and $g(i) = -1$ if $\pi(i) = n$.
(ii) $f|[-i, -i+1]$ is linear for $i \in \{1, \ldots, n-1\}$ and $f|[0, 1] = \pi(0)$. $g|[i, i+1]$ is linear for $i \in \{0, 1, \ldots, n-2\}$ and $g|[-1, 0] = \pi(0)$.

It is easy to show that f (resp. g) is a PSRT type (resp. NSRT type) π with order n, but $\mathrm{Card}(\{x : f(x) = x\}) = \mathrm{Card}(\{x : g(x) = x\}) = 1$. Which implies that π is not turbulent (see Remark 1.3).

In the following, we only study the number of the NSRT types with order n which are not turbulent. Denote by $u_n(k)$ the number of the NSRT types π with order n which are not turbulent with $k = \min\{r : \pi(r) < r\}$ and by $u_n(i, k)$ the number of the NSRT types π with order n which are not turbulent with $k = \min\{r : \pi(r) < r$ and $\pi(0) = i\}$.

Lemma 2.2. *For any $1 \leq r \leq k-1$, we have*

$$u_n(r, k) = \sum_{i=1}^{r} (-1)^{i-1} C_{r-1}^{i-1} u_{n-i}(k-i). \tag{2.1}$$

Proof. If $r = 1$, then

$$u_n(1, k) = u_{n-1}(k-1) = (-1)^0 C_0^0 u_{n-1}(k-1).$$

Assume that (2.1) holds for $1 \leq r < k-1$. That is

$$\sum_{i=r}^{n-1} u_{n-1}(i, k-1) = u_n(r, k) = \sum_{i=1}^{r} (-1)^{i-1} C_{r-1}^{i-1} u_{n-i}(k-i).$$

Then

$$\begin{aligned} u_n(r+1, k) &= \sum_{i=r}^{n-1} u_{n-1}(i, k-1) - u_{n-1}(r, k-1) \\ &= u_n(r, k) - u_{n-1}(r, k-1) \\ &= \sum_{i=1}^{r} (-1)^{i-1} C_{r-1}^{i-1} u_{n-i}(k-i) \\ &\quad - \sum_{i=1}^{r} (-1)^{i-1} C_{r-1}^{i-1} u_{n-1-i}(k-1-i) \\ &= \sum_{i=1}^{r+1} (-1)^{i-1} C_{r}^{i-1} u_{n-i}(k-i). \end{aligned}$$

The proof is completed. □

Write $S_n(k) = \sum_{i=1}^{k-1} u_n(i, k)$. It follows from Lemma 2.2 that

$$\begin{aligned} S_n(k) &= \sum_{i=1}^{k-1} \sum_{j=1}^{i} (-1)^{j-1} C_{i-1}^{j-1} u_{n-j}(k-j) \\ &= \sum_{i=1}^{k-1} (-1)^{i+1} C_{k-1}^{i} u_{n-i}(k-i). \end{aligned}$$