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Stochastic response of a class of self-excited systems with Caputo-type fractional derivative driven by Gaussian white noise

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ABSTRACT

The stochastic response of a class of self-excited systems with Caputo-type fractional derivative driven by Gaussian white noise is considered. Firstly, the generalized harmonic function technique is applied to the fractional self-excited systems. Based on this approach, the original fractional self-excited systems are reduced to equivalent stochastic systems without fractional derivative. Then, the analytical solutions of the equivalent stochastic systems are obtained by using the stochastic averaging method. Finally, in order to verify the theoretical results, the two most typical self-excited systems with fractional derivative, namely the fractional van der Pol oscillator and fractional Rayleigh oscillator, are discussed in detail. Comparing the analytical and numerical results, a very satisfactory agreement can be found. Meanwhile, the effects of the fractional order, the fractional coefficient, and the intensity of Gaussian white noise on the self-excited fractional systems are also discussed in detail.

1. Introduction

The last few decades have witnessed a dramatic development in the research of the fractional calculus partly because of its powerful potential applications in various fields: mechanics, control, biology, chemistry, acoustics, finance, social sciences and especially viscoelastic materials. More and more facts [1–9] show that the fractional-order models can make a more accurate description and give a deeper insight into the inherent nature of realistic physical systems. Thus, many researchers devote themselves to the theoretical analysis and practical application of fractional calculus. A rich variety of excellent books, review articles, papers and monographs dealing with fractional calculus and its application are available [4,8–18].

Because the random factors are ubiquitous in the practical world, it is necessary to study the stochastic systems with fractional derivative. In order to obtain the approximately analytical solution of fractional stochastic systems, a lot of effective approaches have been used by many researchers. The stochastic averaging method which is a versatile and powerful approximate approach has been used by lots of authors. Specifically, Huang and Jin [19] utilized the stochastic averaging method to get the stochastic response and stability of a SDOF stochastic system with fractional derivative damping driven by Gaussian white noise; Chen and Zhu studied the stationary responses [20], stochastic jump and bifurcation [21], stochastic stability [22] and first passage failure [23,24] of stochastic oscillators endowed with fractional derivative damping; Hu and Zhu [25,26] examined the stochastic optimal control of quasi-integrable Hamiltonian

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systems endowed with fractional derivative damping; Xu and Yang [27] studied the stochastic response of stochastic system endowed with Caputo-type fractional derivative damping driven by Gaussian white noise excitation by using the stochastic averaging method. Xu and Zhang [28] discussed the stationary response of Duffing–Rayleigh system with fractional derivative under Gaussian white noise excitation. There are many other techniques to address the stochastic fractional systems except for the stochastic averaging method. Spanos and Zeldin [29] put forward a frequency-domain method for the stochastic systems endowed with fractional derivatives. Based on the machinery of the Wiener path integral, Di Matteo et al. [30] developed a new approximate analytical technique to determine the non-stationary response probability density function of stochastic oscillators with fractional derivatives term. Agrawal [31] proposed an analytical approach for stochastic dynamic systems with fractional derivative by using the eigenvector expansion method and Laplace transforms. Di Paola et al. [32] numerically investigated the stochastic response of a linear viscoelastic system under stationary and non-stationary random excitations by discretizing the fractional derivative operator and increasing the system dimension, in which the key idea has been utilized by Failla and Pirrotta [33] to estimate the stochastic response of fractionally damped Duffing oscillators subjected to a stochastic input. Xu and Li [34,35] put forward a new approach combining the L–P method and the multiple-scale method to obtain the response of stochastic oscillator with fractional derivative. Liu [36] investigated the principal resonance responses of SDOF systems with small fractional derivative damping subjected to narrow-band random parametric excitation by using multiple scale method.

When establishing concrete mathematical models by using the fractional order derivative, researchers are always confronted with two alternatives, namely the Riemann–Liouville definition and the Caputo definition. The reason why we use the Caputo definition in this paper instead of Riemann–Liouville definition is the convenience to obtain the initial conditions. However, many references [19–26], obtaining the approximate analytical solution by using stochastic averaging method, mainly adopt the Riemann–Liouvill definition instead of Caputo definition. Because of the difference between Caputo definition and Riemann–Liouvill definition, the process obtaining the approximate analytical solution is also different. So, when using the stochastic averaging method, it is necessary to study the stochastic systems endowed with Caputo-type fractional derivative.

Bearing these ideas in mind, this paper is organized as follows. In Section 2, the generalized harmonic function technique is applied to the fractional self-excited systems. Based on this approach, the original fractional self-excited systems are reduced to the equivalent stochastic systems without fractional derivative. In Section 3, the analytical solutions of the equivalent stochastic systems are obtained by using the stochastic averaging method. In Section 4, an effective algorithm for the solution of initial value problems with Caputo-type fractional derivative is briefly presented. In Section 5, in order to verify the theoretical results, the two most typical self-excited systems with fractional derivative, namely the fractional van der Pol oscillator and the fractional Rayleigh oscillator, are discussed in detail.

2. Equivalent stochastic system

Consider a class of self-excited oscillators with fractional derivative and subjected to a weak random fluctuation. The motion of the system is governed by the following equation:

$$\begin{aligned}&\ddot{x}(t)+\varepsilon\beta_1 D^{\alpha}x(t)+\varepsilon\beta_2 f(x,\dot{x})\dot{x}+\omega^2 x=W(t)\\&0<\alpha<1\end{aligned}\tag{1}$$

where ε is a small positive parameter, β_1, β_2 and ω are constant coefficients, $W(t)$ is a stationary Gaussian white noise with correlation function $E[W(t)W(t+\tau)]=2D\delta(\tau)$, $f(x,\dot{x})$ is a function of x and $\dot{x}$, $f(x,\dot{x})=\begin{cases}x^2-c \text{ for the fractional van der Pol oscillator}\\ \dot{x}^2-c \text{ for the fractional Rayleigh oscillator}\end{cases}$. There are many definitions for the existence of a fractional derivative. In this paper, the Caputo-type fractional derivative is adopted:

$$D^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\dot{x}(u)}{(t-u)^{\alpha}}du\tag{2}$$

in which α is the fractional order.

According to Refs. [20,37,38], the term associated with fractional derivative not only serves as classical damping force, but also contributes to the restoring force. Based on the generalized harmonic function technique [20], this term can be replaced by the following force containing a linear restoring force and a linear damping force:

$$D^{\alpha}x(t)=C(\alpha)\dot{x}(t)+K(\alpha)x(t)\tag{3}$$

In order to calculate $C(\alpha)$ and $K(\alpha)$, we first introduce two following formulae:

$$\lim_{T\to\infty}\int_0^T\frac{\sin(\omega t)}{t^{\alpha}}=\omega^{\alpha-1}\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}\tag{4}$$

$$\lim_{T\to\infty}\int_0^T\frac{\cos(\omega t)}{t^{\alpha}}=\omega^{\alpha-1}\Gamma(1-\alpha)\sin\frac{\alpha\pi}{2}\tag{5}$$

When ε is a small positive parameter and $W(t)$ represent weakly external random excitation, the response of system (1) can be seen as random spread of periodic solutions of the conservative nonlinear $\ddot{x}(t)+g(x)=0$ around the trivial solution (0, 0) in phase plane $(x,\dot{x})$. Thus, we can assume the solution of system (1) has the following form

$$\begin{aligned}&X=x(t)=A(t)\cos\Phi'(t)\\&Y=\dot{x}(t)=-A(t)\omega\sin\Phi'(t)\\&\Phi'(t)=\omega t+\Theta'(t)\end{aligned}\tag{6}$$

First, we present the detail derivation procedure for $C(\alpha)$.

$$\begin{aligned}C(\alpha)&=-\frac{1}{\pi A\omega}\int_0^{2\pi}D^{\alpha}(A\cos\Phi')\sin\Phi' d\Phi'\\&=\frac{2}{\Gamma(1-\alpha)}\lim_{T\to\infty}\frac{1}{T}\int_0^T\\&\quad\times\left\{\left[\int_0^t\frac{\sin(\omega u+\Phi')}{(t-u)^{\alpha}}du\right]\sin(\omega t+\Phi')\right\}dt\end{aligned}\tag{7}$$