Chaos, Solitons and Fractals 77 (2015) 215–224

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A gradient flow approach to the model of positive feedback in decision-making

Natalia Zabzina*

Department of Mathematics, Uppsala University, Box 480, SE-75106 Uppsala, Sweden

ARTICLE INFO

Article history:
Received 10 February 2015
Accepted 22 May 2015
Available online 14 June 2015

Keywords:
Mathematical model of positive feedback in decision making
Gradient flow equations
Differential geometry,

ABSTRACT

Recent studies on social dynamics have been done by using tools and methods of physics and economics. The main idea is that the regularity observed on a global scale arises out of local interactions between the group members. We consider the model describing one of the major interaction mechanism, the model of positive feedback. We propose a geometrical reformulation of this model in terms of gradient flow equations on a Riemannian manifold. The benefit of this reformulation is that we introduce an alternative method to study phenomena of the well known model. We suggest the analogy with a particle moving on curved manifold. We believe that this analogy will allow us to extend powerful mathematical tools from analytical mechanics to the biological systems.

1. Introduction

During last years it has been renewed interests in the study of the social dynamics using the tools from physics and economics [8,24,34]. The ideas from statistical physics are widely used for the social systems since numerous laws of nature are of statistical origin. Our social behaviour arises out of local interactions with different peers but on a global scale we exhibit regular pattern. This type of emergent behaviour can be analysed by the applications of statistical physics to interdisciplinary fields. The approach to study the social dynamics is to consider the contacts with a limited number of individuals within the population in order to understand the transition from a disordered state to a configuration that displays an order. While borrowed from economics the game theory is now widely applied to the social systems. It has been illustrated how game-theoretic reasoning can help to provide qualitative insight into different forms of biological interaction [17,33]. The key insight of evolutionary game theory (the version of game theory adapted to the biological systems) is that behaviours involve the interaction of multiple organisms in a population and that the success of any one of these organisms depends on how its behaviour interacts with that of others. The competition for the food source can be considered as a two-player game where the fitness that each individual gets from a given food-related interaction can be thought of as a numerical payoff in the two-player game between two individuals. The fitness of an individual in a population is the expected payoff it receives from an interaction with a random member of the population. If each individual is using the best response to what the other individuals are doing and for evolutionary settings this strategy is a genetically-determined strategy then this efficient evolutionary stable strategy tends to persist once it is accepted. Thus we consider uniformities on a global scale. The environment and many other factors can affect the outcome of the evolution strategies. It is demanding to explore successful mechanisms that identify the efficient strategy. Coevolutionary rules constitute a natural upgrade of evolutionary games because the environments and the strategies both evolve in time. The social dilemmas emerge at different levels of human and animal interactions but coevolutionary rules have often not been considered for many game types. For example, in review [32] it has been discussed that coevolution may

* Tel.: +46 184710123.
E-mail address: zabzina@math.uu.se

http://dx.doi.org/10.1016/j.chaos.2015.05.027
0960-0779/

promote cooperation by introducing dynamical mechanisms. We see that huge work has been done to understand the interaction mechanisms and adoptions of different strategies depending on the opponents which lead to a particular decision-making. The big step towards realistic conditions is the recent shift from evolutionary games on regular grids to evolutionary games on complex networks [1,16,17,26].

The modelling of social interaction sets aside a huge number of details but the main concept in many social dynamic models is the idea that in order to interact two individuals must be alike. There are rules of thumb in the context of social dynamic in group decision-making and the most important of these is a positive feedback between group members [36]. Positive feedback is a interaction mechanism which occurs when commitment to a particular option increases as a function of the number of individuals already committed to it [2,6,13,14]. The mathematical model of positive feedback has been proposed and tested experimentally for ant species using trail-laying recruitment [3,4,12,18,39]. The main analytical results for this non-linear model are related to the study of steady states solutions, their existence and stability [27,29,30]. The only physically acceptable solutions (i.e., real and positive solutions) are considered to test the stability of these different branches in order to determine the state that will actually be chosen by the system. The selection of the option depends strongly on the size of the group and on an initial preferences, which will entrain the majority of individuals to focus on a particular option.

The goal of our paper is to apply a geometrical interpretation of this model to highlight an alternative method to study the decision-making process. In particular we suggest the reformulation of the model in terms of gradient flow equations on Riemannian manifold with non-trivial metric. We reinterpret the variables as coordinates on the smooth manifold with the concrete Riemannian metric. The gradient flow equations on Riemannian manifolds can be thought of as continuous versions of the famous steepest descent method for a potential function. The steepest descent method is a first-order optimisation algorithm which allows us to find a local minimum of a potential. In this method one takes steps proportional to the negative of the gradient of the potential function at the current point and gradually moves towards the minimum of potential function. We consider the continuous version of steepest descent method, which is known as gradient flow equations. Thus the corresponding dynamical system is intimately linked with the problem of minimising potential. The gradient flow equations are widely used in modern geometry and optimisation theory. In our treatment of gradient flow we closely follow the following books [20] and [37].

The benefits of the present geometrical reformulation of the model are: first, by investigating a graph of smooth potential function we introduce another method to study certain qualitative and quantitative aspects of this non-linear model. The solution of the model evolves in such way that the potential function should decrease along the solution and as $t \to \infty$ the solution approaches the equilibrium solution which corresponds to the minimum of potential. Moreover this picture manifests some of the crucial biological features of the system. For example, the bias to initial conditions and the amplification of response can be seen very clearly in the gradient flow picture. Another benefit of our reformulation is that it brings to this biological system the powerful mathematical machinery with many geometrical tools in Riemannian geometry and topology. We can apply the existence theorems for the solutions, consider properties of Riemannian manifold and discuss the properties of potential. Those geometrical tools are used actively in the physics and therefore we can use some analogy to physical models of particles moving on Riemannian manifold in the potential. However the potential function is rather exotic compared with the standard physical systems. We hope that this point has great promise for further development and analysis of biological non-linear models. As an example we show how the geometrical tools in Riemannian geometry and topology can be used to explore biological phenomena described by this model of positive feedback.

The paper is organised as follows: in Section 2 we review the model and present its reformulation of the model in terms of gradient flow on Riemannian manifolds. Also we provide a short introduction to the gradient flow equations. Section 3 is devoted to the application of gradient flow to biological system. Section 3.1 illustrates the approach for the case of two options and Section 3.2 deals with the analysis of the results and suggest some analogy with physical systems. In last the section we give the summary and discussion of the results.

2. Gradient flow reformulation of the model of positive feedback

We consider the model which describes positive feedback loops and each loop measures how the buildup of commitment to a particular option evolves in time. We denote by q_i the associated "quality" of each option and assume that commitment decays at a constant rate ν. The evolution of the commitments c_i to option i can be expressed in the form

$$\frac{dc_i}{dt} = \phi q_i f_i(c_1, \ldots, c_s) - \nu c_i, \qquad i = 1, \ldots, s, \tag{1}$$

where s is the number of the options and ϕ is the ant flow from the nest towards the options, related to the colony size. Here ϕ is assumed to be a constant. The choice function $f_i(c_1, \ldots, c_s)$ expresses how future commitment to an option is affected by the current commitment and is here assumed to take the form

$$f_i(c_1, \ldots, c_s) = \frac{(c_i + k)^l}{\sum_{j=1}^{s}(c_j + k)^l}. \tag{2}$$

The parameter k is the threshold number beyond which the choice of option begins to be effective and l stands for the sensitivity of the response to the process of choice of a particular option. Thus we write the model of positive feedback in the form

$$\frac{dc_i}{dt} = \phi q_i \frac{(c_i + k)^l}{\sum_{j=1}^{s}(c_j + k)^l} - \nu c_i, \qquad i = 1, \ldots, s. \tag{3}$$

Throughout the paper we use the following conventions $\phi q_i = h_i$. We are interested in solving these ODEs with prescribed initial conditions $c_i(0)$. Moreover due to the interpretation of c_i as commitment (e.g., the concentration of