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A note on the points with dense orbit under the expansions of different bases

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ABSTRACT

It was conjectured by Furstenberg that for any $x \in [0,1]\backslash\mathbb{Q}$,

$$\dim_H \overline{\{2^n x (\bmod 1) : n \geq 1\}} + \dim_H \overline{\{3^n x (\bmod 1) : n \geq 1\}} \geq 1,$$

where $\dim_H$ denotes the Hausdorff dimension and $\overline{A}$ denotes the closure of a set A. When x is a normal number, the above result holds trivially. In this note, we are aiming at giving explicit non-normal numbers for which the above dimensional formula holds.

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1. Introduction

Let $b \geq 2$ be an integer. Use T_b to denote the classic b-adic transformation:

$$T_b(x) = bx - \lfloor bx \rfloor, \quad x \in [0,1),$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Then every x in the unit interval $[0, 1)$ can be developed uniquely into a finite or infinite series

$$x = \sum_{k \geq 1} \frac{i_k(x,b)}{b^k} = 0. \quad i_1 i_2 \ldots,$$

where the digits $i_1, i_2, \ldots$ are integers from $\{0, 1, \ldots, b-1\}$ and they form the expansion of x in base b.

Almost all x in the unit interval $[0, 1)$ (according to Lebesgue measure) are normal to base b or, equivalently, the sequences $\{T_b^n(x)\}_{n \geq 1}$ are uniformly distributed [8]. To some extent (in the sense of Lebesgue measure), the expansion of any point is clear under the transformation T_b for any integer $b \geq 2$. However, the actual situation is that when two transformations T_{b_1}, T_{b_2} are taken into consideration simultaneously, one can obtain very few information about the b_1-adic expansion of x by knowing its b_2-adic expansion [4]. It is widely believed that if the expansion of x under T_{b_1} is 'simple', it cannot be too 'simple' under T_{b_2}. For example, if the b_1-adic complexity of x is low, then the b_2-adic complexity of x would be high [4]. Because of such a phenomenon, Furstenberg [6] conjectured that: for any $x \in [0,1]\backslash\mathbb{Q}$,

$$\dim_H \overline{\{2^n x (\bmod 1) : n \geq 1\}} + \dim_H \overline{\{3^n x (\bmod 1) : n \geq 1\}} \geq 1,$$

where the notation $\dim_H$ denotes the Hausdorff dimension and $\overline{A}$ denotes the closure of a set A.

Furstenberg's conjecture holds for almost all real numbers, because almost all of them are normal to all integer bases, hence they are normal to base 2 and base 3. So, a natural question is to ask, besides normal numbers, can one give explicit examples fulfilling the above conjecture? In this short note, we construct a set of real numbers that are not normal but Furstenberg's conjecture holds.

Theorem 1. *There exists a Cantor set E composed of real numbers in the unit interval such that*

1: *Each x in E is not normal to base* 2.
2: *For each* $x \in E$, $\overline{\{2^n x (\bmod 1) : n \geq 1\}} = [0,1]$ *and* $\overline{\{3^n x (\bmod 1) : n \geq 1\}} = [0,1]$.
3: $\dim_H E = 1$.

The relation between expansions of a point under different bases was studied for a long history in the literature at

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least up to Maxfield [9] in 1953. Cassels [5] and Schmidt [11] proved the existence of points which are normal to one base but not normal to another base. Our idea to construct numbers which are not normal follows from Schmidt [11]. Similar idea was used recently to construct normal numbers, see Becher, Heiber & Slaman [1] and Becher & Slaman [2]. For the normality of points on fractals, see the recent progress made by Hochman & Shmerkin [7]. A good reference for these topics is Bugeaud's book [4], which contains more than 700 references.

To end this section, we give some evidences that the expansion of a point may be 'simple' under one transformation but not too 'simple' under the other.

- Denote by $\mathcal{DC}$ the number of digit changes, i.e.,

  $$\mathcal{DC}(x,n,b) = \text{Card}\{1 \le k \le n : i_k(x,b) \ne i_{k+1}(x,b)\},$$

  where Card denotes the cardinality of a finite set. Let b_1, b_2 be coprime integers larger than 2. Bugeaud [4, Theorem 6.8] proved that there exist an integer n_0 and a positive real number κ such that

  $$\mathcal{DC}(x,n,b_1) + \mathcal{DC}(x,n,b_2) \ge \kappa \log n, \quad \textit{for } n > n_0.$$

  This means that if the b_1-adic expansion of x has few digit changes, then the b_2-adic expansion of x must have a certain amount of digit changes.
- Recall that the ternary Cantor set $\mathcal{C}$ consists of points whose digits in 3-adic expansions contain only 0,2. But as proved by Schmidt [10], there exist uncountably many points in $\mathcal{C}$ which are normal to base 2.
- This last item may not fit our situation quite well, but it is a concrete example: the continued fraction expansion of $\sqrt{2}$ is very 'simple'

  $$\sqrt{2} = [1; 2, 2, \ldots],$$

  however we even have no idea about the occurrence of the number 7 in its decimal expansion.

2. Some notation

The following is a list of notations that will be used later.

- A word: a finite sequence of symbols.
- $\mathcal{C}_1$ and $\mathcal{C}_2$: the collection of all finite 2-adic and 3-adic words respectively, i.e.

  $$\mathcal{C}_1 = \bigcup_{n\ge1}\{0,1\}^n, \quad \mathcal{C}_2 = \bigcup_{n\ge1}\{0,1,2\}^n. \tag{2.1}$$

- $\mathcal{A} = (w_1, v_1, w_2, v_2, \ldots)$: an enumeration of the words in $\mathcal{C}_1$ and $\mathcal{C}_2$ where $w_i \in \mathcal{C}_1$ and $v_i \in \mathcal{C}_2$, and, for each $i \ge 1$, w_i apears infinitely many times, and v_i too.
- $|w|$: the length of the word w.
- 0^m: a word of length m composed of 0.
- $[w]_2$: a 2-adic cylinder of order $|w|$, i.e.

  $$[w]_2 := \{x \in [0,1) : \text{the 2-adic expansion of } x \text{ starts with } w\}.$$

- $[v]_3$: a 3-adic cylinder defined similarly.

For the sake of completeness, we recall the definition of normality for real numbers. For any integer $b \ge 2$, let

$$x = \frac{i_1(x,b)}{b} + \frac{i_2(x,b)}{b^2} + \ldots,$$

so $i_1(x,b)i_2(x,b)\ldots$ is the b-adic expansion of $x \in [0,1)$. For each finite word $D \in \bigcup_{n\ge1}\{0,1,\ldots,b-1\}^n$, write

$$N(x,b,n,D) = \text{Card}\Big\{1 \le k \le n : (i_k(x,b), \ldots, i_{k+|D|-1}(x,b)) = D\Big\}$$

i.e., the number of the occurrence of the block D in the first n digits of the b-adic expansion of x.

Definition 1. A point $x \in [0,1)$ is called normal to base b, if for each $D \in \bigcup_{n\ge1}\{0,1,\ldots,b-1\}^n$,

$$\lim_{n\to\infty} \frac{N(x,b,n,D)}{n} = \frac{1}{b^{|D|}}.$$

Otherwise, it is not normal. Call x absolutely normal if it is normal to all bases $b \ge 2$.

3. Proof of the main results

The technique of the proof is the construction of a nested sequence of intervals controlling the base-2 and base-3 expansions of the real numbers in these intervals. We illustrate the technique as a lemma.

Lemma 1. *Let b_1, b_2 be two integers greater than or equal to 2. Let $[i_1,\ldots,i_n]_{b_1}$ be a b_1-adic cylinder of order n. Then there exists a b_2-adic cylinder $[j_1,\ldots,j_k]_{b_2}$ such that*

$$[j_1,\ldots,j_k]_{b_2} \subset [i_1,\ldots,i_n]_{b_1},$$
$$2b_2^{-k} \le b_1^{-n} < 2b_2^{-k+1}.$$

Proof. Let k be the integer such that

$$2b_2^{-k} \le b_1^{-n} < 2b_2^{-k+1}.$$

The first inequality on the choice of k implies that the length of the cylinder $[i_1,\ldots,i_n]_{b_1}$ is larger than twice of that of a b_2-adic cylinder of order k. So, it must contain one b_2-adic cylinder of order k. Denote such a b_2-adic cylinder by $[j_1,\ldots,j_k]_{b_2}$ (if there are many, just choose one), which fulfills the desired conditions. □

Such a plain observation was formulated in the work of Schmidt [11], where he showed the existence of points normal to a collection of bases, but not normal to other bases. The same idea was also used in [1] to construct numbers which are normal to all integer bases.

The desired set E will be a union of a collection of Cantor sets $\{E_m\}_{m\ge2}$, each of which satisfies the first two items in Theorem 1, and whose Hausdorff dimensions tend to 1 as $m \to \infty$.

Before the construction, we fix some sequences of integers. Recall that $\mathcal{A} = \{w_1, v_1, w_2, v_2, \ldots\}$ is an enumeration of the elements in $\mathcal{C}_1$ and $\mathcal{C}_2$ such that each element in $\mathcal{C}_1 \cup \mathcal{C}_2$ appears infinitely many times.

For each $k \ge 1$, let $d_k = |w_k| + (|v_k| + 1)\log_2 3 + 3$. Choose a sequence of positive integers $\{\ell_k\}_{k\ge1}$ such that

$$\lim_{k\to\infty} \frac{d_1 + \cdots + d_k}{\ell_k} = 0. \tag{3.1}$$

Fix an integer $m \ge 2$ and put $n_k = m \cdot \ell_k$ for all $k \ge 1$. Recall that $\mathcal{C}_1^n$ denotes the collection of 2-adic words of length n

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