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Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations

Kamal Shah*, Hammad Khalil, Rahmat Ali Khan

Department of Mathematics, University of Malakand, Chakadara Dir(L), Khyber Pakhtunkhwa, Pakistan

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ABSTRACT

In this article, we study a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations. We obtain sufficient conditions for existence and uniqueness of positive solutions. We use the classical fixed point theorems such as Banach fixed point theorem and Krasnoselskii's fixed point theorem for uniqueness and existence results. As in application, we provide an example to illustrate our main results.

1. Introduction

Fractional order differential equations have attracted the attention of many researchers because of their applications in many scientific and engineering disciplines such as physics, chemistry, biology, viscoelasticity, control theory, signal and image processing phenomenons, economics, bioengineering etc, we refer to [1–5,7,11–15] and the references therein. It is investigated that fractional order differential equation model real world problems more accurately than differential equations of integral order.

Recently, the study of existence and uniqueness of solutions to initial and boundary value problems for fractional order differential equations has attracted considerable attention, we refer to [6–9,32] and the references therein. The study of coupled systems of fractional order differential equations has also attracted some attention. Because mathematical models of various phenomenona in the field of physics, biology and psychology etc., are in the form of coupled systems of differential equations. For the study of coupled systems of fractional order differential equations, we refer to [16–20]. Another important class of differential equations is known as impulsive differential equations. This class plays the role of an effective mathematical tools for those evolution processes that are subject to abrupt changes in their states. There are many physical systems that exhibit impulsive behavior such as the action of a pendulum clock, mechanical systems subject to impacts, the maintenance of a species through periodic stocking or harvesting, the thrust impulse maneuver of a spacecraft, and the function of the heart, we refer to [25] for an introduction to the theory of impulsive differential equations. It is well known that in the evolution processes the impulsive phenomena can be found in many situations. For example, disturbances in cellular neural networks [28], operation of a damper subjected to the percussive effects [26], change of the valve shutter speed in its transition from open to closed state [27], fluctuations of pendulum systems in the case of external impulsive effects [29], percussive systems with vibrations [30], relaxational oscillations of the electromechanical systems [31], dynamic of system with automatic regulation [32], control of the satellite orbit, using the radial acceleration [32] and so on. The theory of impulsive differential equations is well studied and the large number of research articles are available in the literature on impulsive differential equations, we refer to [5,10–15] and the references therein for some of the recent development in the theory.

* Corresponding author. Tel.: +923463025197.
E-mail addresses: kamalshah408@gmail.com (K. Shah), hammadk310@gmail.com (H. Khalil), rahmat_alipk@yahoo.com (R.A. Khan).

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Recently, Feng et al. [21] studied existence of positive solutions to the following impulsive boundary value problem (IBVP)

$$\begin{cases} -(\phi_p(u'(t)))' = f(t,u(t)), \quad t\in[0,1], \quad t\neq t_k, \\ \quad k=1, \quad 2\dots n, \\ -\Delta u(t_k) = I_k(y(t_k)), \quad k=1,2,\dots,n, \\ u'(0)=0, \quad u(1)=\int_0^1 g(t)u(t)dt. \end{cases}$$

Wang et al. [22] developed a sufficient condition for existence of positive solutions to the following impulsive boundary value problem (IBVP) via topological degree theory

$$\begin{cases} {}^cD^q u(t) = f(t,u(t)), \quad t\in[0,T]\setminus D := \{t_1,t_2,\dots,t_m\}, \\ u(0)=0, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i=1,2,\dots,m, \end{cases}$$

where $0<q<1$ and $I_i:\mathbb{R}\to\mathbb{R}$ are nonlinear functions describing the jump size $(I_i(u(t_i)) = u(t_i^+)-u(t_i^-))$ at t_i, $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$. Significant development has been made in last few years in the theory of impulsive fractional order differential equations with fixed moments. It gives natural description of observed evolution process and is an important tool for understanding several real world phenomenona. Y.Tian, et al. [23] developed some interesting results for the existence of positive solutions to the following impulsive fractional order differential equations (IFBVP)

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \quad t\in[0,1], \quad t\neq t_k, \quad k=1,2\dots p, \\ \Delta x(t_k) = I_k(x(t_k)), \quad \Delta x'(t_k) = \bar{I}_k(x(t_k)), \quad k=1,2,\dots,p, \\ x(0)=g(x), \quad x(1)=h(x), \end{cases}$$

where $1<q\le 2$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous, $I_k,\bar{I}_k:\mathbb{R}\to\mathbb{R}$ are continuous, and $\Delta(x(t_k)) = x(t_k^+)-x(t_k^-)$, $\Delta(x'(t_k)) = x'(t_k^+)-x'(t_k^-)$ with $x(t_k^+), x'(t_k^+), x(t_k^-), x'(t_k^-)$, are the respective left and right limits of $x(t_k)$ at $t=t_k$. Zhang et al. [24] extended the work to coupled system of 2m-point BVP for impulsive fractional differential equations at resonance as:

$$\begin{cases} D^\alpha_{0+}u(t) = f(t,v(t),D^p v(t)), \quad D^\beta_{0+}v(t) = g(t,u(t), \\ \quad D^q u(t))\, t\in(0,1), \\ \Delta u(t_i) = A_i(v(t_i), \quad D^p v(t_i)), \quad \Delta u(t_i) = B_i(v(t_i), \\ \quad D^q v(t_i)), \quad i=1,2,\dots k; \\ \Delta v(t_i) = C_i(u(t_i), \quad D^q u(t_i)), \quad \Delta v(t_i) = D_i(u(t_i), \\ \quad D^q u(t_i)), \quad i=1,2,\dots k; \\ D^{\alpha-1}u(0) = \sum_{i=1}^m a_i D^{\alpha-1}u(\xi_i), \quad u(1) = \sum_{i=1}^m b_i \eta_i^{2-\alpha}u(\eta_i); \\ D^{\beta-1}v(0) = \sum_{i=1}^m c_i D^{\beta-1}v(\varsigma_i), \quad v(1) = \sum_{i=1}^m d_i\theta_i^{2-\beta}v(\theta_i), \end{cases}$$

where $1<\alpha,\beta<2$, $\alpha-q\ge 1$, $\beta-p\ge 1$.

Motivated by the importance of the study mentioned above of impulsive fractional differential equations, we develop sufficient conditions for existence and uniqueness of solutions to the following complex dynamical network in the form of a coupled system of $m+2$-point boundary conditions for impulsive fractional differential equations

$$\begin{cases} {}^cD^\alpha u(t) = \Phi(t,u(t),v(t)), \quad t\in[0,1], \quad t\neq t_j, \\ \quad j=1,2,\dots,m, \\ {}^cD^\beta v(t) = \Psi(t,u(t),v(t)), \quad t\in[0,1], \\ \quad t\neq t_i, \quad i=1,2,\dots,n, \\ u(0)=h(u), \quad u(1)=g(u) \text{ and } v(0)=\kappa(v), \\ \quad v(1)=f(v), \\ \Delta u(t_j) = I_j(u(t_j)), \quad \Delta u'(t_j) = \bar{I}_j(u(t_j)), \\ \quad j=1,2,\dots,m, \\ \Delta v(t_i) = I_i(v(t_i)), \quad \Delta v'(t_i) = \bar{I}_i(v(t_i)), \\ \quad i=1,2,\dots,n, \end{cases} \tag{1}$$

where $1<\alpha,\ \beta\le 2$, $\Phi,\ \Psi:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions and $g,\ h:X\to\mathbb{R}$, $f,\ \kappa:Y\to\mathbb{R}$ are continuous functionals define by

$$g(u) = \sum_{j=1}^p \lambda_j u(\xi_j), \quad h(u) = \Sigma_{j=1}^p \lambda_j u(\eta_j),$$

$$f(v) = \sum_{i=1}^q \delta_i v(\xi_i), \quad \kappa(v) = \sum_{i=1}^q \delta_i v(\eta_i),$$

$\xi_i, \eta_i, \xi_j, \eta_j\in(0,1)$ for $i=1,2,\dots,p$, $j=1,2,\dots,q$, and

$$\Delta u(t_j) = u(t_j^+)-u(t_j^-), \quad \Delta u'(t_j) = u'(t_j^+)-u'(t_j^-)$$
$$\Delta v(t_i) = v(t_i^+)-v(t_i^-), \quad \Delta v'(t_i) = v'(t_i^+)-v'(t_i^-).$$

As given above, the notations $u(t_j^+), v(t_i^+)$ are right and $u(t_j^-), v(t_i^-)$ are left limits, respectively. Moreover, $I_r, \bar{I}_r(r=j,i):\mathbb{R}\to\mathbb{R}$ are continuous functions and D^α_{0+}, D^β_{0+} stand for Caputo's fractional derivative of order α, β, respectively. We assume that $\sum_{r=1}^p \lambda_j\eta_r^{\alpha-1}<1$, $\sum_{r=1}^q \delta_r\xi_r^{\alpha-1}<1$, $r=i,j$. Here, we remark that the system (1) includes as a special case the following well-studied Hopfield neural networks and the design model in the presence of impulses [28]

$$\begin{cases} \dfrac{du_i}{dt} = -a_i u_i(t) + \sum_{j=1}^m b_{ij} f_j(u_j(t)) + c_i, \quad t\neq t_k, \\ \quad i=1,2,\dots,m, \quad k=1,2,\dots, \\ \Delta u_i(t_k) = I_i(u_i(t_k)). \end{cases}$$

We obtain necessary and sufficient conditions for the existence and uniqueness of positive solution via Krasnoselskii's fixed point theorem and Banach contraction principle. We give an example to illustrate our main results.

2. Background materials

In this section, we recall some basic definitions and results from fractional calculus, fixed point theory and functional analysis [1–5,33,34].

Definition 2.1. The Rieman–Liouvilli fractional integral of order $\alpha\in\mathbb{R}_+$ of a function $\omega\in C((0,\infty),\mathbb{R})$ is defined as

$$I^\alpha_{0+}\omega(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\omega(s)\,ds,$$

where $\alpha>0$ and Γ is the gamma function, provided that the right side is point wise defined on $(0,\infty)$.