Chaos, Solitons and Fractals 77 (2015) 277–285

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A computation method for non-autonomous systems with discontinuous characteristics

Yuu Miino [a,*], Daisuke Ito [b], Tetsushi Ueta [c]

[a] Advanced Technology and Science, System Innovation Engineering, Tokushima University, 2-1 Minami-Josanjima, Tokushima 770-8506, Japan
[b] Department of Electronic Systems Engineering, School of Engineering, The University of Shiga Prefecture, 2500 Hassaka-cho, Hikone, Shiga 522-8533, Japan
[c] Center for Admininistration Information Technology, Tokushima University, 2-1 Minami-Josanjima, Tokushima 770-8506, Japan

ARTICLE INFO

Article history:
Received 28 February 2015
Accepted 15 June 2015
Available online 5 July 2015

Keywords:
Bifurcation phenomena
Numerical analysis
Nonlinear non-autonomous system
Discontinuity

ABSTRACT

We propose a computation method to obtain bifurcation sets of periodic solutions in non-autonomous systems with discontinuous properties. If the system has discontinuity for the states and/or the vector field, conventional methods cannot be applied. We have developed a method for autonomous systems with discontinuity by taking the Poincaré mapping on the switching point in the preceded study, however, this idea does not work well for some non-autonomous systems with discontinuity. We overcome this difficulty by extending the system to an autonomous system. As a result, bifurcation sets of periodic solutions are solved accurately with a shooting method. We show two numerical examples and demonstrate the corresponding laboratory experiment.

1. Introduction

To analyze stability or bifurcations for a periodic solution of an ordinary differential equation (ODE) with smooth nonlinear characteristics, a bunch of computational packages or algorithms are available [1–3]. Basically these methods convert the periodic motion into a fixed point problem by taking Poincaré map and solve it by applying an appropriate shooting method.

On the other hand, if the system contains non-smooth properties, some special treatments should be considered since continuousness of the map for the given ODE is lost [4–6], fortunately some non-smooth systems can be analyzed by putting an approximated smooth function into the ODE. Otherwise, defining a composite Poincaré map with multiple sections attached to break points is required [7]. How to construct the differentiable Poincaré map whose characteristic equation results correct multipliers is a key point of these methods. Previous study [8] showed bifurcation analysis of behaviors on the stepping motor which has an autonomous dynamical system and has discontinuity on its vector field. Another one [9] showed global bifurcation analysis of such systems. However, as far as we know, there are no bifurcation analysis for non-autonomous systems with discontinuity, thus we focus on these systems. In fact, if we apply Kousaka's method which is for autonomous systems to these non-autonomous systems, the shooting method converges slowly and leaves some errors. Thereby tracing bifurcation sets sometimes fails because of accumulation of these errors. We intuitively guess that some special treatment for adding a forcing term in evaluation of the Jacobian matrix.

At first, we tried applying previously mentioned method [7] simply to the forced Izhikevich neuron model [10], which is a non-autonomous system with discontinuity, but then we obtained the differentiable Poincaré map with some errors.

* Corresponding author. Tel.: +717055141303.
E-mail addresses: c501437041@tokushima-u.ac.jp, nagamonnyuu@gmail.com (Y. Miino), ito.d@e.usp.ac.jp (D. Ito), ueta@tokushima-u.ac.jp (T. Ueta).

http://dx.doi.org/10.1016/j.chaos.2015.06.014

The error could not be ignored to analyze bifurcation structure of the system.

In this paper, we investigate a cause of the errors and propose a universal algorithm for solving bifurcation problems of nonlinear non-autonomous hybrid systems. The remaining contents are organized as follows; Section 2 describes the problem formulation. Section 3 investigates what is a cause of the errors with numerical experiments and propose our idea to solve it. Section 4 devotes to show validation of our method with two examples, i.e., we discuss agreement among computed bifurcation diagrams, numerical solutions, and laboratory experiments of each system. Section 5 concludes our study.

2. Numerical analysis of nonlinear non-autonomous system with discontinuous characteristic

As proposed on the previous study [11], when we consider systems with discontinuous characteristics, we often define the Poincaré section with the condition of discontinuity. However on smooth non-autonomous systems, we often define the Poincaré section with the time because periodic solutions have a periodicity synchronized with the frequency of the external force. Thus on this paper we define the Poincaré section of the systems with the time and try to analyze it by previously mentioned method.

Similarly to the previous study [12], let us consider an n-dimensional non-autonomous system with m-tuple differential equations described by

$$\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \boldsymbol{R}^n, \tag{1}$$

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}_i(t, \boldsymbol{x}), \quad i = 0, \ldots, m-1, \tag{2}$$

where $t \in \boldsymbol{S}^1$ is the adjusted time with $\boldsymbol{S}^1 = \{t \in \boldsymbol{R} \mod \tau\}$, $\tau \in \boldsymbol{R}$, which is often $2\pi/\omega$, is a parameter for an initial section Π_0, $\boldsymbol{x} \in \boldsymbol{R}^n$ is the state and $\boldsymbol{f}_i$: $\boldsymbol{R}^n \to \boldsymbol{R}^n$ is a C^∞ class function.

Assume that there is a periodic solution for Eq. (2). When we suppose that Π_i is a traversal section to the solution orbit and put $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \Pi_0$, then the solution of Eq. (2) is given by

$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t, \boldsymbol{x}_0). \tag{3}$$

Moreover, each solution following $\boldsymbol{f}_i$ is given by $\boldsymbol{\varphi}_i(t, \boldsymbol{x}_i, t_i)$, where t_i is the starting time of the solution. Now we provide Π_i with threshold values as follows:

$$\Pi_i = \left\{t \in \boldsymbol{S}^1, \boldsymbol{x} \in \boldsymbol{R}^n | q_i(t, \boldsymbol{x}, \lambda_i) = 0\right\}, \tag{4}$$

where q_i is a differentiable scalar function and $\lambda_i \in \boldsymbol{R}$ is a unique parameter that defines the position of Π_i. In addition, on the non-autonomous system,

$$\Pi_0 = \left\{t \in \boldsymbol{S}^1, \boldsymbol{x} \in \boldsymbol{R}^n | q_0(t, \boldsymbol{x}, \tau) = t = 0\right\}. \tag{5}$$

When an orbit governed by $\boldsymbol{f}_i$ reaches the section Π_{i+1}, the governing function is changed to $\boldsymbol{f}_{i+1}$. If the orbit passing through several sections reaches Π_0 again, then m sub maps are defined as follows:

$$\begin{array}{lll} T_0 & : & \Pi_0 \to \Pi_1 \\ & & \boldsymbol{x}_0 \mapsto \boldsymbol{x}_1 = \boldsymbol{\varphi}_0(t_1, \boldsymbol{x}_0, t_0 = 0) \\ T_1 & : & \Pi_1 \to \Pi_2 \\ & & \boldsymbol{x}_1 \mapsto \boldsymbol{x}_2 = \boldsymbol{\varphi}_1(t_2, \boldsymbol{x}_1, t_1) \\ & & \vdots \\ T_{m-1} & : & \Pi_{m-1} \to \Pi_0 \\ & & \boldsymbol{x}_{m-1} \mapsto \boldsymbol{x}_m = \boldsymbol{\varphi}_{m-1}(t_m, \boldsymbol{x}_{m-1}, t_{m-1}). \end{array} \tag{6}$$

From Eq. (6), the Poincaré map T is given by the following composite map:

$$T(\boldsymbol{x}(k), \tau, \lambda_1, \ldots, \lambda_{m-1}) = T_{m-1} \circ \cdots \circ T_1 \circ T_0. \tag{7}$$

Hence

$$\boldsymbol{x}(k+1) = T(\boldsymbol{x}(k), \tau, \lambda_1, \ldots, \lambda_{m-1}). \tag{8}$$

When the orbit starting from $\boldsymbol{x}_0 \in \Pi_0$ returns $\boldsymbol{x}_0$ itself, this orbit forms a periodic orbit and the corresponding fixed point of T is written as follows:

$$\boldsymbol{x}_0 = T(\boldsymbol{x}_0, \tau, \lambda_1, \ldots, \lambda_{m-1}). \tag{9}$$

The characteristic equation is given by

$$\chi(\mu) = \det\left(\frac{\partial T(\boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} - \mu I\right) = 0, \tag{10}$$

where μ is a multiplier of $\partial T(\boldsymbol{x}_0)/\partial \boldsymbol{x}_0$. When the multiplier satisfies $|\mu| = 1$, solution attractors of the system occurs bifurcation phenomena. In other words, μ can be given as $|\mu| = 1$ to obtain a bifurcation parameter set.

3. Problem and solution idea

3.1. Problem on numerical analysis

Here we found a problem on numerical analysis previously mentioned. In common for the systems with discontinuous characteristics, we could use composition of maps for $\partial T/\partial \boldsymbol{x}_0$ as:

$$\frac{\partial T}{\partial \boldsymbol{x}_0} = \prod_{i=0}^{m-1} \left.\frac{\partial T_{m-1-i}}{\partial \boldsymbol{x}_{m-1-i}}\right|_{t_{m-1-i}}^{t_{m-i}}. \tag{11}$$

However on numerical experiment, right hand of Eq. (11) is not equal to its left hand of it. Now we confirm this with an example of 1-periodic orbit observed in forced Izhikevich neuron model introduced in Section 4.1, see Fig. 1.

By using numerical differentiation, we can obtain $\partial T_i/\partial \boldsymbol{x}_i$ roughly:

$$\frac{\partial T_0}{\partial \boldsymbol{x}_0} = \begin{pmatrix} 0.00000 & 0.00000 \\ -0.05423 & 0.70627 \end{pmatrix}, \tag{12}$$

$$\frac{\partial T_1}{\partial \boldsymbol{x}_1} = \begin{pmatrix} 0.00000 & 0.00000 \\ 1.00000 & 1.00000 \end{pmatrix}, \tag{13}$$

$$\frac{\partial T_2}{\partial \boldsymbol{x}_2} = \begin{pmatrix} -0.99281 & 0.36780 \\ 0.24401 & -0.09478 \end{pmatrix}, \tag{14}$$

and for this example, we can regard $m = 3$ and then Eq. (11) is expressed by