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Some peculiarities of the solution of the heat conduction equation in fractional calculus

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ABSTRACT

The solution of the heat conduction equation in derivatives of fractional order with the account of diffuse and convective mechanisms of heat transfer is provided. The dependence of the temperature distribution on the rates of derivatives of fractional order by time and coordinate is studied.

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1. Introduction

Intensive development of both fundamental and applied aspects of mathematics of fractional calculus is started since the systems with the fractional structure were discovered [1]. It appeared that the study of both equilibrium and non-equilibrium processes in system having fractal structure requires application of the fractional integrals and derivatives. In paper [2] was given the physical interpretation of the fractional integral. The integral $f(t) = \int_0^t g(t-\tau)h(\tau)d\tau$ where $h(t)$ – input signal, $f(t)$ – output signal and $g(t)$ – memory function, was considered. The absence of memory corresponds to the case $g(t) = \delta(t)$ where δ – Dirac delta function. The case $g(t) = \Theta(t)$, where $\Theta(t)$ – Heaviside step function, corresponds to the presence of full memory. The author of paper [2] wondered how will change the expression for the $f(t)$ if $g(t)$ occupies an intermediate position between the functions $\delta(t)$ and $\Theta(t)$, that is nonzero value on the fractional set. For the set of fractional order the Cantors set is chosen. Initial time interval $[0, t]$ is divided into three segments with the length ξt, $(\xi < 1/2)$ and the middle segment is removed. Remaining segments with length ξt are processed by the same procedure. This procedure is repeated infinitely remaining the value of initial integral constant. In result of limiting transition the initial integral is converted to the integral of fractional order $f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1}(\tau)d\tau,\ \nu \in (0, 1]$. However, in papers [3,4] it was found that such transition is mathematically incorrect and there is no direct relationship of fractional integral with the Cantors set. Further in paper [5] the relationship between stable distribution of the probability theory and the fractional integral was defined. Random value X that is defined by characteristic function of the form $f(t) = exp(-a^{\alpha}|t|^{\alpha})$, ($a \geqslant 0,\ \alpha \in (0, 2]$ - stability parameter) has stable density of distribution equal to the Fourier transform of the characteristic function. The value X is explicitly defined for two cases: for $\alpha = 2$ we have centered normal law $p(x) = \frac{1}{\sqrt{4\pi a}} exp\left(-\frac{x^2}{4a}\right)$, and for $\alpha = 1$ we have Cauchy law with density $p(x) = \frac{a}{\pi}\frac{1}{x^2+a^2}$. The paper [5] shows that appearance of fractional derivative in time is connected with the random process having stable distribution of probability and parameter of stable distribution coincide with the rate of derivative of fractional order. In fact the necessity of application of the fractional calculus for the study of non-equilibrium properties of systems with fractional structure is associated with the fundamental physical aspects. The peculiarity of the systems with fractional structure is the existence of well-developed interphase boundary, that is actually require to use the geometry of fractional structure

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that has self-similarity [1]. The portion of the interphase boundary is relatively big that contributes to the observing characteristics of matter. Let us note that interphase boundary is the special state of matter, that occupies an intermediate position between the existing phases, and its nature at present time is not quite clear [6]. Particularly, relaxation processes on the interphase boundary have complex character that leads to a non-linear and non-local processes of heat-and-mass transfer. As a result the principle of local equilibrium fails. It is required to use the principle of local non-equilibrium. That eventually lead to appearance of the different generalizations of thermodynamics [7]. The non-local effects account is generally performed by means of integral operator. The integral operator kernel contains the nature of non-locality. To solve the integro-differential equation it is required to reduce the integral operators to the series of differential operators with the ascending differentiation rates. Obtained equations appear to be too complex and not always can be solved. In this context the technique of integro-differentiation of fractional order is the subject of interest. It is currently considered that the mathematical technique of fractional calculus [8–10] presents a new approach to the description of the transfer processes in systems for which it is significant to account the non-local properties in time and by coordinate. Unlike the traditional equations of mathematical physics the fractional calculus contains additional parameters rates of the derivatives of fractional order that leads to the emergence of a new class of solutions. Using the rates of derivatives of fractional order as parameters for explanation experimental data we get new information. Thus, in paper [11] on the base of equation in derivatives of fractional order in time, analyzing known experimental data of kinetics and sorption were obtained fundamentally new results. In work [12] an integral-balance method to solve fractional sub-diffusion equations has been conceived. Due to using the rates of derivatives of fractional order in interpretation of experimental data, the study of relation between the solution of the heat conduction equation in derivatives of fractional order and the rates of the fractional derivatives is the subject of interest.

2. Heat equation in derivatives of fractional order with convective transfer mechanism

In research of heat transfer in complex heterophase structures are widely used equations of parabolic type in derivatives of fractional order [13,14].

$$\frac{\partial^\alpha T(\xi,\tau)}{\partial\tau^\alpha} - D\frac{\partial^\beta}{\partial\xi^\beta}T(\xi,\tau) + V\frac{\partial^\gamma}{\partial\xi^\gamma}T(\xi,\tau) = 0 \tag{1}$$

Included in (1) derivatives of fractional order are defined as follows

$$\frac{\partial^\alpha T(\xi,\tau)}{\partial\tau^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_0^\tau \frac{T(\xi,\tau)}{(\tau-z)^\alpha}dz - \frac{T(\xi,0)}{\Gamma(1-\alpha)\tau^\alpha} \tag{2}$$

$$\frac{\partial^\beta T(\xi,\tau)}{\partial\xi^\beta} = \frac{1}{2\Gamma(2-\beta)\cos(\pi(2-\beta)/2)}\frac{\partial^2}{\partial\xi^2} \times \int_{-\infty}^{\infty}\frac{T(\xi',\tau)}{|\xi-\xi'|^{\beta-1}}d\xi' \tag{3}$$

$$\frac{\partial^\gamma T(\xi,\tau)}{\partial\xi^\gamma} = \frac{1}{2\Gamma(1-\gamma)\cos(\pi(1-\gamma)/2)}\frac{\partial}{\partial\xi} \times \int_{-\infty}^{\infty}\frac{T(\xi',\tau)}{|\xi-\xi'|^\gamma}d\xi' \tag{4}$$

where $|\xi| < \infty, \tau > 0$;

$T(\xi,\tau)$ – temperature;
$\tau = t/t_0,\ \xi = x/x_0$ – non-dimensional time and coordinate;
$D = at_0/x_0^2$ – non-dimensional thermal diffusivity;
$a = \lambda/c_p\rho$ – thermal diffusivity;
c_p – specific isobaric heat capacity, ρ – density;
$t_0,\ x_0$ – characteristic time and typical scale;
$0 < \alpha \leqslant 1,\ 0 < \gamma \leqslant 1,\ 1 < \beta \leqslant 2$;
$V = vx_0/t_0$ – non-dimensional speed;
v – velocity of convective stream of matter.

Caputo derivative (2) accounts memory (non-locality in time), Riss derivative (3) and (4) accounts spatial correlation (spatial non-locality), $\Gamma(x)$ – Euler gamma function. To solve the Eq. (4) we use the Fourier transform by the space variable

$$T_F(k,\tau) = \int_{-\infty}^{\infty} exp(-ikx)T(x,\tau)dx$$

and the Laplace transform by the time variable

$$T_{FL}(k,p) = \int_0^\infty exp(-p\tau)T_F(k,\tau)d\tau$$

Performing the Fourier and the Laplace transform, we obtain the following expression

$$T_{FL}(k,p) = \frac{T_F(k,0)}{p^{1-\alpha}(p^\alpha + D|k|^\beta - iV|k|^\gamma sign(k))} \tag{5}$$

Performing an inverse Fourier and Laplace transform, we have

$$T(\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp(-ik(\xi-\xi'))T(\xi',0) \times E_{\alpha,1}(-(D|k|^\beta - iV|k|^\gamma sign(k))\tau^\alpha)dk\, d\xi' \tag{6}$$

Here $E_{\alpha,\nu}(-z^\alpha) = \sum_{n=0}^{\infty}(-1)^n\frac{z^{\alpha n}}{\Gamma(\alpha n+\nu)}$ – Mittag–Lefflers function [8]. Solution (6) depends of three parameters α, β, γ and as it shown below in particular cases corresponds to earlier known solutions.

3. Analysis of the obtained solutions

To analyze the dependence of the solution (6) on the parameters α, β and γ consider the case for $T(\xi,0) = T_0\delta(\xi)$. Then from (6) we obtain

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