



Explicitly defined fractal interpolation functions with variable parameters



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ABSTRACT

We construct an explicit formula for the fractal interpolation function associated to an IFS with variable parameters. The solution is given in terms of the base p representation of numbers. This construction is a consequence of the formulation of the problem in a general functional equation setting. We introduce compatibility conditions as essential hypotheses to ensure problems in the functional system form are well-defined.

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1. Introduction

Given a set of data

$$\Delta = \{(x_i, y_i) \in I \times J \subset \mathbb{R}^2 : i = 0, 1, 2, \dots, N\}, \quad (1)$$

where $x_0 < x_1 < \dots < x_N$ is a partition of the interval $I = [x_0, x_N]$, Barnsley [1] constructed fractal interpolation functions (FIF) considering the associated iterated function system $\{ \mathbb{R}^2; w_n, n = 1, 2, \dots, N \}$, where the maps w_n are affine transformations with the special form

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix},$$

constrained by the data according to

$$w_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \text{ and } w_n \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad n = 1, 2, \dots, N.$$

This system of affine transformations has one free parameter which is chosen to be d_n . All other parameters a_n, c_n, e_n, f_n are determined by d_n .

The problem of finding a fractal interpolation function can be formulated in terms of functional relations, as

illustrated by Barnsley and Harrington [2]. Their setting for computing FIFs is the following.

Let Δ be as defined in (1). Let L_j be the affine map satisfying

$$L_j(x_0) = x_{j-1}, \quad L_j(x_N) = x_j, \quad j = 1, 2, \dots, N. \quad (2)$$

Let $-1 < \alpha_j < 1$ and $F_j : I \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions such that, for $j = 1, 2, \dots, N$,

$$|F_j(x, \xi_1) - F_j(x, \xi_2)| \leq |\alpha_j| |\xi_1 - \xi_2|, \quad x \in I, \quad \xi_1, \xi_2 \in \mathbb{R}$$

and

$$F_j(x_0, y_0) = y_{j-1}, \quad F_j(x_N, y_N) = y_j. \quad (3)$$

As shown in [2], the FIF associated with $\{(L_j(x), F_j(x, y))\}_{j=1}^N$ is the unique function $f : I \rightarrow \mathbb{R}$ satisfying

$$f(L_j(x)) = F_j(x, f(x)), \quad j = 1, 2, \dots, N. \quad (4)$$

The classical setting studied in [2] is the special case where $F_j(x, y) = \alpha_j y + q_j(x)$. Wang and Yu [22] recently introduced a class of IFSs with variable parameters which generates fractal interpolation functions (FIFs). Using a normalized interval $I = [0, 1]$, the problem is generalized by allowing the parameters α_j to depend on x :

$$F_j(x, y) = \alpha_j(x)y + q_j(x). \quad (5)$$

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Fractal interpolation functions are a very useful and efficient way to construct models that approximate data exhibiting irregular structure. Many areas of the applied sciences explore this interpolation method numerically, modeling in general with linear IFS. We mention for instance areas as Medicine, with studies about cardiac arrhythmia [5], tumor perfusion [7], electroencephalograms [18] and neural networks [19]; Physics or Engineering, with works on turbulence [3,20], wind speed prediction [25], Hydrology [16], Seismology [12,23], speech signals [21] and PN waveform generators [9]; Biology or Economics, with studies on DNA or stock prices [11,24]. FIFs can be related with other topics within Mathematics such as estimation of fractal dimension [15] or reproducing kernel Hilbert spaces [4].

Much of the research performed on FIF is devoted to the construction of algorithms, numerical simulations and evaluations of algebraic and analytic properties (e.g. study of fractal dimension, stability or smoothness). We present here an explicit functional form of the FIF with variable parameters in terms of base p representation of numbers. In order to perform this we construct solutions of systems of functional equations of type (4). The study of these systems is not complete and is the subject of ongoing research (see, e.g. [26]).

2. Systems of functional equations

Let X and Y be non-empty sets and $p \geq 2$ an integer. Consider a system of functional equations

$$\varphi(f_j(x)) = F_j(x, \varphi(x)), \quad x \in X_j, \quad j = 0, 1, \dots, p-1, \quad (6)$$

where $X_j \subset X$, $f_j: X_j \rightarrow W_j \subset X$, $F_j: X_j \times Y \rightarrow Y_j \subset Y$ are given functions, and $\varphi: \bigcup_{j=0}^{p-1} X_j = X \rightarrow Y$ is the unknown function.

If each F_j does not depend explicitly on x , i.e., the system (6) is of the form

$$\varphi(f_j(x)) = F_j(\varphi(x)), \quad x \in X_j, \quad j = 0, 1, \dots, p-1,$$

then the equations of system (6) are conjugacies.

Proposition 1. *If $\varphi: X \rightarrow Y$ is a solution of (6), then it must satisfy*

$$\begin{aligned} \forall x \in X_i, \quad \forall y \in X_j, f_i(x) = f_j(y) \\ \Rightarrow F_i(x, \varphi(x)) = F_j(y, \varphi(y)) \end{aligned} \quad (7)$$

for $i, j = 0, 1, \dots, p-1$.

Proof. Suppose $x \in X_i$, $y \in X_j$ are such that $f_i(x) = f_j(y)$. By (6) we have

$$\begin{cases} \varphi(f_i(x)) = F_i(x, \varphi(x)), \\ \varphi(f_j(y)) = F_j(y, \varphi(y)). \end{cases} \quad (8)$$

Equality of the left-hand sides of (8) then implies that

$$F_i(x, \varphi(x)) = F_j(y, \varphi(y)),$$

as stated. $\square$

We call conditions (7) the **compatibility conditions** for the system (6), since they ensure that the function φ is well-defined when the images of different X_j intersect.

Let $A := \{x \in X : \exists y \in X, \exists i, j = 0, 1, \dots, p-1, i \neq j, f_i(x) = f_j(y)\}$. We call the elements of A **contact points** of the system (6). Given a system of Eq. (6), once the images of φ for the contact points are determined (by partially solving the system or by initial conditions of the problem), compatibility conditions are necessary conditions on the f_i and F_j for the existence of solutions, as shown by the following example.

Example 2. Let $X = [0, 1]$ and consider the system of functional equations

$$\begin{cases} \varphi(f_0(x)) = F_0(x, \varphi(x)), & x \in [0, 1], \\ \varphi(f_1(x)) = F_1(x, \varphi(x)), & x \in [0, 1]. \end{cases} \quad (9)$$

Suppose

$$f_0(x) = \frac{x}{2}, \quad f_1(x) = \frac{x+1}{2}.$$

Here $X_0 = X_1 = [0, 1]$. Since

$$f_0(X_0) \cap f_1(X_1) = \left\{ \frac{1}{2} \right\}$$

and

$$f_0(1) = \frac{1}{2} = f_1(0),$$

the compatibility condition is

$$F_0(1, \varphi(1)) = F_1(0, \varphi(0)). \quad (10)$$

If we suppose, additionally, that $F_i(x, y) = \alpha_i(x)y + q_i(x)$, then the compatibility condition is

$$\alpha_0(1)\varphi(1) + q_0(1) = \alpha_1(0)\varphi(0) + q_1(0).$$

Solving the first equation of (9) for $x=0$ and the second for $x=1$, we obtain for the images of the contact points

$$\varphi(0) = \frac{q_0(0)}{1 - \alpha_0(0)}, \quad \varphi(1) = \frac{q_1(1)}{1 - \alpha_1(1)}.$$

The compatibility condition on F_0, F_1 is

$$\frac{\alpha_0(1)q_1(1)}{1 - \alpha_1(1)} + q_0(1) = \frac{\alpha_1(0)q_0(0)}{1 - \alpha_0(0)} + q_1(0).$$

The autonomous version of this problem (that is, where F_0, F_1 do not depend explicitly on x) was studied by de Rham [17], who showed existence and uniqueness of solution for the corresponding system of functional equations under the assumption $F_0(\varphi(1)) = F_1(\varphi(0))$ which corresponds to (10) in this special case. More general cases were studied, for instance, by Girgensohn [10] and by Zdun and Ciepliński [6,26].

The following result of Girgensohn has the advantage of giving an explicit solution in terms of the base p representation of numbers. However, it still does not provide the solution to the more general problem defined by the system of functional equation (4) with variable parameters (5).

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