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Time-dependent iteration of random functions

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ABSTRACT

In studying the iteration of random functions, the usual situation is to assume time-homogeneity of the process and some average contractivity condition. In this paper we change both of these conditions by investigating the iteration of time-dependent random functions where all the functions converge (as the iterations proceed) uniformly to the identity. The behaviour of the iterates is remarkably different from the standard contractive situation. In particular, we show that for affine maps in $\mathbb{R}^d$ the "chaos game" trajectory converges almost surely. This is in stark contrast to the usual situation where the trajectory moves ergodically throughout the attractor.

1. Introduction

Iterating random functions is a topic which has received a considerable amount of recent attention [1,5,6,8,12–16]. One reason for this interest is the fact that many Markov chains can be represented as an iteration of random functions (see [8,15]). Furthermore, asymptotic convergence results along with convergence rates can often be obtained by viewing a process as a random iteration. The survey "Iterated Random Functions" [5] contains a very nice discussion of some general theory and many applications. There is also a strongly related literature from the area of random dynamical systems (a nice introduction is [10] and especially [9, Ch. 8] as a quick introduction to random nonautonomous discrete-time dynamical systems).

One fruitful technique in the study of the iteration of random functions is comparing the "forward iterates"

$$w^n_{\sigma_n} \circ w^{n-1}_{\sigma_{n-1}} \circ w^{n-2}_{\sigma_{n-2}} \circ \cdots \circ w^2_{\sigma_2} \circ w^1_{\sigma_1} \tag{1}$$

with the "reverse iterates"

$$w^1_{\sigma_1} \circ w^2_{\sigma_2} \circ w^3_{\sigma_3} \circ \cdots \circ w^{n-1}_{\sigma_{n-1}} \circ w^n_{\sigma_n}. \tag{2}$$

In the time-homogeneous case, the reverse and forward iterations lead to sequences of random variables with the same distribution. This can give simple conditions for the existence of an equilibrium measure as the reverse iteration often converges pathwise to a point which is independent of the starting distribution (at least under an average contractivity condition).

In most of the previous work, it is assumed that the sequence of functions is a time-homogeneous sample and convergence results usually depend on both this assumption and some average contractivity assumption. While there has been some consideration of time-dependent iteration, the iteration is assumed to be asymptotically contractive as well (see [14]) and the results are more delicate. In this paper, we consider the situation where the functions are time-varying and all of them asymptotically converge uniformly to the identity, so are not asymptotically contractive. Surprisingly this is a situation which seems to have been overlooked in the IFS literature and the results we obtain are strikingly different from the standard case of time-homogeneous contractive IFS.

A simple example motivated the study and illustrates our results quite well. Take a decreasing positive sequence $a_n \to 0$ with $a_1 < 1/2$ and define the sequence of (time-dependent) maps

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$$w_0^n(x) = (1 - a_n)x, \quad w_1^n(x) = (1 - a_n)x + a_n. \tag{3}$$

In addition, choose $p_0 \in [0, 1]$ and $p_1 = 1 - p_0$ to be fixed probabilities. Then $w_i^n(x) \to x$ as $n \to \infty$ for $i = 0, 1$, each w_i^n is contractive, $w_0^n(0) = 0$, and $w_1^n(1) = 1$. However, the behaviour of both the forward and reverse iterates is very different from that of a time-independent iteration and is completely contingent on whether the series $\sum_n a_n$ converges.

If $\sum_n a_n < \infty$, then the "chaos game" (the forward iterates) always converges for any starting point and any sequence of choices of w_0^n or w_1^n. However, the limit can depend on both the starting point and the sequence of maps which was chosen. If $\sum_n a_n = \infty$, the chaos game can also converge, but in this case the limit is almost surely independent of the starting point (or distribution). This is totally different from the classical case where the chaos game never stabilizes. A special case of (3) is nice enough to be worth a mention.

Proposition 1. *Let $a_n = 1/(n+1)$. Choose $x_0 \in [0, 1]$ and each $\sigma_n \in \{0, 1\}$ according to the probabilities p_0, p_1. Using these, define $x_n = w_{\sigma_n}^n(x_{n-1})$. Then $x_n \to p_1$ for all choices of x_0 and almost all choices of σ_n.*

Proof. First, we rewrite the two IFS maps as

$$w_0^n(x) = \frac{nx}{n+1}, \quad w_1^n(x) = \frac{nx+1}{n+1}.$$

Consider a sequence of biased coin flips with $Prob(tails) = p_0$ and $Prob(heads) = p_1$. If we let

$$y_n = \frac{\text{number of heads in the first } n \text{ flips}}{n}$$

we see that, given a flip of a tail on iteration n we have $y_{n+1} = w_0^n(y_n)$ and given a flip of a head on iteration n we have $y_{n+1} = w_1^n(y_n)$. By the Law of Large Numbers, $y_n \to p_1$ with probability one. Thus, $x_n \to p_1$ with probability one as well. $\square$

In fact, as we will show in Theorem 4, as long as $\sum_n a_n = \infty$ the forward iterates of (3) always converge to p_1 almost surely. Theorem 4 is an extension of this idea to arbitrary affine functions in $\mathbb{R}^d$ and is the main result of this paper.

For the reverse iterations of (3), if $\sum_n a_n < \infty$ then the "address map" (the reverse iteration) converges to a limit which depends on the sequence of maps and can depend on the choice of starting point. If $\sum_n a_n = \infty$, then the reverse iteration converges to a limit which depends on the sequence of maps but not the starting point, thus defining an "address map" in the usual way.

2. Preliminaries

We use the language of iterated function systems (IFS) [7,2,11] and first review some of the basic facts of this theory. An IFS on a metric space $\mathbb{X}$ is a finite collection of contractive self-maps $w_i : \mathbb{X} \to \mathbb{X}, i = 1, \ldots, N$; let $c < 1$ be the maximum of the contractivity factors. The *attractor* of the IFS $\{w_i\}$ is the unique nonempty compact set $A \subseteq \mathbb{X}$ with

$$A = \bigcup_i w_i(A). \tag{4}$$

Let $\Sigma = \{1, 2, \ldots, N\}$ so that $\Sigma^{\mathbb{N}}$ is the *code space* of all sequences from Σ and Σ^n are those sequences of length n. We endow $\Sigma^{\mathbb{N}}$ with the metric

$$d(\sigma, \alpha) = \sum_n \frac{|\sigma_n - \alpha_n|}{(N+1)^n}.$$

This metric gives the product topology on $\Sigma^{\mathbb{N}}$ (with the discrete topology on each factor) and makes $\Sigma^{\mathbb{N}}$ into a compact space. Define the truncation of $\sigma \in \Sigma^{\mathbb{N}}$ by $\sigma^n = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma^n$.

The relation (4) can be iterated to get

$$A = \bigcup_{\sigma \in \Sigma^n} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(A). \tag{5}$$

Since $w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \circ w_{\sigma_{n+1}}(A) \subseteq w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(A)$ and also $diam(w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(A)) \leqslant c^n diam(A)$ it is sensible to take the limit of (5) as $n \to \infty$. In fact, for a fixed $\sigma \in \Sigma^{\mathbb{N}}$, the limit

$$\lim_n w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(A) \tag{6}$$

exists and defines the *address map a*: $\Sigma^{\mathbb{N}} \to \mathbb{X}$ whose range is A. Notice the order of composition in (6) is what we have called the reverse iteration.

An iterated function system with probabilities (IFSP) is a collection of IFS maps w_i along with associated probabilities p_i. Letting $\mathcal{P}(\mathbb{X})$ denote the space of Borel probability measures on $\mathbb{X}$, an IFSP induces a Markov operator $M : \mathcal{P}(\mathbb{X}) \to \mathcal{P}(\mathbb{X})$ defined by

$$M\mu(B) = \sum_i p_i\, \mu(w_i^{-1}(B))$$

for an arbitrary Borel subset $B \subset \mathbb{X}$. It is customary to metrize $\mathcal{P}(\mathbb{X})$ using the Monge–Kantorovich metric, given by

$$d_M(\mu, \nu) = \sup\left\{ \int_{\mathbb{X}} f(x)\, d(\mu(x) - \nu(x)) : |f(x) - f(y)| \leqslant d(x, y) \right\}.$$

It is well-known that this metric yields the topology of weak convergence of measures and that $\mathcal{P}(\mathbb{X})$ is compact when $\mathbb{X}$ is compact (see [17, Chap 7] for more on the Monge–Kantorovich metric). If c_i is the contractivity factor of w_i, then M is Lipschitz on $\mathcal{P}(\mathbb{X})$ with factor $\sum_i p_i c_i$. Thus when the IFSP is *average contractive*, $\sum_i p_i c_i < 1$, there is a unique invariant distribution μ which satisfies

$$\mu = M\mu = \sum_i p_i\, \mu \circ w_i^{-1}.$$

In terms of integrals, this is

$$\int_{\mathbb{X}} f(x)\; dM\mu(x) = \sum_i p_i \int_{\mathbb{X}} f(x)\; d\mu(w_i^{-1}(x))$$
$$= \int_{\mathbb{X}} \sum_i p_i f(w_i(y))\; d\mu(y).$$

The forward iteration of the maps w_i are used in the so-called "chaos game" which is a random walk on $\mathbb{X}$ defined as follows. First, we choose a starting point $x_0 \in \mathbb{X}$ in some