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Fractals as objects with nontrivial structures at all scales

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ABSTRACT

Toward the middle of 2001, the authors started arguing that fractals are important when discussing the operational resilience of information systems and related computer sciences issues such as artificial intelligence. But in order to argue along these lines it turned out to be indispensable to define fractals so as to let one recognize as fractals some sets that are very far from being self similar in the (usual) metric sense. This paper is devoted to define (in a loose sense at least) fractals in ways that allow for instance all the Cantor sets to be fractals and that permit to recognize *fractality* (the property of being fractal) in the context of the information technology issues that we had tried to comprehend. Starting from the meta-definition of a fractal as an "object with non-trivial structure at all scales" that we had used for long, we ended up taking these words seriously. Accordingly we define fractals in manners that depend both on the structures that the fractals are endowed with and the chosen sets of structure compatible maps, i.e., we approach fractals in a category-dependent manner. We expect that this new approach to fractals will contribute to the understanding of more of the fractals that appear in exact and other sciences than what can be handled presently.

1. Introduction

(I) – Toward the middle of 2001, we (i.e., both authors and a few other friends) began to try convince some of their colleagues in the information technology industry that *fractals are needed to describe essential aspects of operational resilience (or OR) of computer systems. Computer systems* are big entities that may comprise the business environment when the computers and other machinery are integrated in a big enterprise, and may in particular comprise part at least of the work force. The main obstacle that we met in this task that we set to ourselves – for us a pre-requisite to attack seriously the issue of OR and other important issues of computer technology and computer science such as *artificial intelligence* (or *AI*) – was the fact that most people, at least those out of the inner circle of fractals experts, had a view of fractals that was way too restrictive to let them recognize some computer systems and other objects as fractals.

In this paper we propose to define fractals in a way that depends on the structures that one chooses to equip either the space in which the fractal leaves or the fractal itself and on the chosen set of allowed maps for said structures. One can then say that the dependence is upon *categories*, the formal concept developed to deal with structures, and structure compatible maps. The implications of our work on fractals upon OR, AI, and operational risk, i.e., our original goal, will be reported elsewhere [43]: at best will we allow ourselves some brief remarks in the present paper.

(II) – In Section 2, we discuss briefly as first examples the Koch curve and three families of related objects. We construct these first objects using a method that we call the *"rescale-and-replace construction"* that is based on the same sort of symbolic space that arise in the discussion

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of *Iterated Function Systems* (or *IFSs*) that we use in that same section to present many further examples. We examine briefly a collection of examples of fractals, including Cantor sets as examples of complicated objects that can be constructed using diverse methods. We also introduce in 2 a few structure-dependent forms of *self similarity*, including a hierarchy of more and more general definitions of self similarity that are related to IFSs as another (and more usual) method to construct fractals that are self similar in some sense. We have tried to stay far enough from classical introductions to fractals in order to keep some experts amused, at least till we come to the new material (that appears mostly in 4.), but also with the idea in mind that many non-experts have read some classical material on fractals and would also prefer to start from a new angle. The presentation of the material lets us first manipulate in 2 the idea of *"dependence upon structure"* in the context of *self similarity* that is a concept much easier to grasp (for most structures) than the general concept of "category-dependent fractals" whose study we initiate here.

(III) – In order to illustrate the absence (believed to be the rule) of *strict metric self similarity* (i.e., self similarity that involves only finitely many scaling ratios) in some at least of the fractals that one is faced with in natural sciences, we report in 3 on the very essential absence of strict metric self similarity of the Universal period doubling Cantor set (that we present with enough details and background to justify the "natural" character that we attribute to that object). We expound without getting into the details of the proof, that *the ratios that define the generic universal period doubling attractor (for smooth enough generic unimodal maps) (or GUPDA) accumulate on another universal Cantor set*, the *generically universal asymptotic ratios Cantor set* (or *GUARCS*). A map is *unimodal* if its graph has a single turning point. This result from [10] answered positively in 2003 a 1977 conjecture by Coullet and Tresser [15]. We will also discuss the way various ratios that form the GUARCS are located on the GUPDA in order to have a fuller view on these very special Cantor sets. These results from [10] are presented here for the first time in a manner that is only mildly technical. *To our knowledge, no other deterministic fractal has so far showed up in experiments and exhibited a set of scaling ratios proved to be infinite* despite the generally accepted fact that *such complicated geometric structure is the rule rather than the exception for fractals that emerge when studying natural phenomena.* The level of precision that we obtain on the GUPDA is in particular too fine to possibly be fully captured by so called *multi-fractal analysis*, a point a view also defended for instance by Feigenbaum [22]. Of course this does not prevent multi-fractal analysis from being an important field. The name *multi-fractal analysis* (and more precisely the prefix 'multi") is what we criticize in 3 as creating confusion about the role of strict metric self similarity in fractality. This is because strict metric self similarity is so rare in nature that it is rather pathological in natural sciences and other applications. This statement is both about strict deterministic self similarity and *strict statistical self similarity* (by which we mean random objects such that the rescaled versions of any of pieces of these objects at different metric scales do not appear as depending on the scale). Strictly metrically self similar fractals do appear in some mathematical contexts or and man made objects.

(IV) – The proposed structure-dependent definitions of fractals (or more precisely the category-dependent definitions as explained there) are formulated in 4. There we also discuss various consequences of using the definitions of fractals that we propose (we have many definitions precisely because there are many different categories). As a striking example, with a definition of topological fractals at hand, we give examples of fractals which, beyond lack of metrical self similarity, also fails strongly to be topologically self similar in that no two points have topologically similar (i.e., homeomorphic) neighborhoods: we do not know how frequent is such a severe lack of topological self similarity, neither in mathematics nor in any natural science, business or technology: we suspect a great abundance. Although we propose definitions we have done our best to not be too formal; we are not aiming at having the ultimate point of view. *This paper is mostly an invitation to a cross-disciplinary collective effort.*

2. Starting with basic vocabulary and a few examples

2.1. Preliminary mathematics

The definitions and basic properties of general foundational concepts, e.g., about topological and metric spaces are easily found in many *world wide web* (or *www*) resources. We provide here and then all along as needed, the definitions that we consider as most essential for the discussion.

2.1.1. Distances

The map $d : Q^2 \to \mathbb{R}^+$ is a *distance* on Q if for any $x,\ y,\ z$ in Q:

- $d(x,y) = 0$ if and only if $x = y$,
- $d(x,y) \equiv d(y,x)$,
- $d(x,z) \leqslant d(x,y) + d(y,z)$.

A *metric space* is a set endowed with a distance.

The *length* of a segment (relative to d) is the distance between its extremities. We assume the plane, and more generally $\mathbb{R}^n$ for $n \geqslant 1$, equipped with the Euclidean distance d_E given when $n = 2$ (the case on which we shall concentrate first) by:

$$d_E((x_0, y_0), (x_1, y_1)) = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

for any pair of points $((x_0, y_0), (x_1, y_1))$ in some orthonormal basis. Let (X, d) be a *complete* metric space (for the usual distance, the set $\mathbb{Q}$ of rational numbers is not a complete set and $\mathbb{R}$ is the smallest complete space containing $\mathbb{Q}$: see also the www and references found there). Let then $\mathcal{C}(X)$ stand for the set of non-empty subsets of X that are *compact* (i.e., closed and bounded subsets when dealing with complete metric spaces). We write $B(c, \rho)$ for the set of points x in (X, d) such that $d(c, x) \leqslant \rho$, i.e., the closed ball