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Review

Conservative flows with various types of shadowing

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ABSTRACT

In the present paper we study the C^1-robustness of the three properties: average shadowing, asymptotic average shadowing and limit shadowing within two classes of conservative flows: the incompressible and the Hamiltonian ones. We obtain that the first two properties guarantee dominated splitting (or partial hyperbolicity) on the whole manifold, and the third one implies that the flow is Anosov.

1. Introduction: basic definitions and statement of the results

It is known since long time ago that nonlinear systems behave, in general, in a quite complicated fashion. One of the most fundamental example of that was given by Anosov when studying the geodesic flow associated to metrics on manifolds of negative curvature [3]. Anosov obtained a striking geometric-dynamical behavior, now called *uniform hyperbolicity*, of those systems in particular a global form of uniform hyperbolicity (a.k.a. Anosov flows). The core characteristic displayed by uniform hyperbolicity is, in brief terms, that on some invariant directions by the tangent flow we observe uniform contraction or expansion along orbits, and these rates are uniform. In the 1960s the hyperbolicity turned out to be the main ingredient which trigger the construction of a very rich theory of a wide class of dynamical systems (see e.g. [47,35]). It allows us to obtain a fruitful geometric theory (stable/unstable manifolds), a stability theory (in rough terms that hyperbolicity is tantamount to structural stability), a statistical theory (smooth ergodic theory) and a numerical theory (shadowing and expansiveness) are some examples of powerful applications of the uniform hyperbolicity concept. However, from an early age one began to understand that the uniform hyperbolicity was far from covering all types of dynamical systems and naturally other more relaxed definitions began to emerge (nonuniform hyperbolicity, partial hyperbolicity and dominated splitting, see e.g. [23]).

As mentioned above, the hyperbolicity was found to contain very interesting numerical properties. Actually, the hyperbolic systems display the *shadowing property*: meaning that quasi-orbits, that is, almost orbits affected with a certain error, were shaded by true orbits of the original system. This amazing property, which is not present in partial hyperbolicity (see [22]), contained itself much of the typical rigidity of the hyperbolicity and its strong assumptions. Nonetheless, a much more surprising fact is that, under a certain stability hypothesis, the other way around turns out to be also true. To be more precise, if we assume that we have the robustness of the shadowing property, then the dynamical system is uniformly hyperbolic. In overall, some stability of a pointedly numerical property, allow us to obtain a geometric, dynamic and also topological feature. The next step then was to address the following question: *is it possible to weaken the shadowing property and obtain the same conclusions?* If not, *how far can we get in our findings?*

In the present paper we deal with three enfeebled branches of shadowing: the average shadowing, the asymptotic average shadowing and the limit shadowing (see

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Section 1.3 for full details). In conclusion, we prove that the stability of these types of shadowing for conservative flows imply (some) hyperbolicity. More specifically, the stability of the first two types of shadowing mentioned above imply that the flow admits a dominated splitting in the whole manifold, and the one of the third shadowing guarantees that the flow is of Anosov type. Theses results hold for incompressible flows and Hamiltonian ones and in arbitrarily high dimension. See Section 1.4 for the statements of the main results of this work.

1.1. Dissipative and incompressible flows setting

Along this paper we consider vector fields $X : M \to TM$, where M is a d-dimensional ($d \geqslant 3$) connected and closed C^∞ Riemannian manifold M and TM its tangent bundle. Given a vector field X we have an associated flow X^t which is the infinitesimal generator of X in a sense that $\partial_t X^t|_{t=s}(p) = X(X^s(p))$. If the divergence of X, defined by $\nabla \cdot X = \sum_{i=1}^{d} \frac{\partial X_i}{\partial x_i}$, is zero we say that X is divergence-free. The flow X^t has a tangent flow DX_p^t which is the solution of the non-autonomous linear variational equation $\partial_t u(t) = DX_{X^t(p)} \cdot u(t)$. Moreover, due to Liouville's formula, if X is divergence-free, the associated flow X^t preserves the volume-measure and for this reason we call it *incompressible*. If the vector field is not divergence-free its flow is *dissipative*. We denote by $\mathfrak{X}^1(M)$ the set of all C^1 vector fields and by $\mathfrak{X}^1_\mu(M) \subset \mathfrak{X}^1(M)$ the set of all C^1 vector fields that preserve the volume, or equivalently the set of all incompressible flows. We assume that both $\mathfrak{X}^1(M)$ and $\mathfrak{X}^1_\mu(M)$ are endowed with the C^1 Whitney (or strong) vector field topology which turn these two vector spaces completed, thus a Baire space. We denote by $\mathcal{R}$ the set of regular points of X, that is, those points x such that $X(x) \neq \vec{0}$ and by Sing $(X)= M \setminus \mathcal{R}$ the set of singularities of X. Let us denote by Crit(X) the set of critical orbits of X, that is, the set formed by all periodic orbits and all singularities of X.

The Riemannian structure on M induces a norm $\|\cdot\|$ on the fibers T_pM, $\forall p \in M$. We will use the standard norm of a bounded linear map L given by

$$\|L\| = \sup_{\|u\|=1} \|L(u)\|.$$

A metric on M can be derived in the usual way by using the exponential map or through the Moser volume-charts (cf. [40]) in the case of volume manifolds, and it will be denoted by $d(\cdot,\cdot)$. Hence, we define the open balls $B(x,r)$ of the points $y \in M$ satisfying $d(x,y) < r$ by using those charts.

Dissipative flows appear often in models given by differential equations in mathematical physics, economics, biology, engineering and many diverse areas. Incompressible flows arise naturally in the fluid mechanics formalism and has long been one of the most challenging research fields in mathematical physics.

1.2. The Hamiltonian flow formalism

Let (M,ω) be a compact symplectic manifold, where M is a $2d$-dimensional ($d \geqslant 2$), smooth and compact Riemannian manifold endowed with a symplectic structure ω, that is, a skew-symmetric and nondegenerate 2-form on the tangent bundle TM. We notice that we use the same notation for manifolds supporting Hamiltonian flows and also flows as in Section 1.1, which we hope will not be ambiguous.

We will be interested in the Hamiltonian dynamics of real-valued C^2 functions on M, constant on each connected component of the boundary of M, called *Hamiltonians*, whose set we denote by $C^2(M,\mathbb{R})$. For any Hamiltonian function $H : M \longrightarrow \mathbb{R}$ there is a corresponding *Hamiltonian vector field* $X_H : M \longrightarrow TM$, tangent to the boundary of M, and determined by the equality

$$\nabla_p H(u) = \omega(X_H(p), u), \quad \forall u \in T_pM,$$

where $p \in M$.

Observe that H is C^2 if and only if X_H is C^1. Here we consider the space of the Hamiltonian vector fields endowed with the C^1 topology, and for that we consider $C^2(M,\mathbb{R})$ equipped with the C^2 topology.

The Hamiltonian vector field X_H generates the *Hamiltonian flow* X_H^t, a smooth 1-parameter group of symplectomorphisms on M satisfying $\partial_t X_H^t = X_H(X_H^t)$ and $X_H^0 = id$. We also consider the *tangent flow* $D_pX_H^t : T_pM \longrightarrow T_{X_H^t(p)}M$, for $p \in M$, that satisfies the linearized differential equality $\partial_t D_pX_H^t = (D_{X_H^t(p)}X_H) \cdot D_pX_H^t$, where $D_pX_H : T_pM \longrightarrow T_pM$.

Since ω is non-degenerate, given $p \in M$, $\nabla_p H = 0$ is equivalent to $X_H(p) = 0$, and we say that p is a *singularity* of X_H. A point is said to be *regular* if it is not a singularity. We denote by $\mathcal{R}$ the set of regular points of H, by Sing (X_H) the set of singularities of X_H and by Crit (H) the set of critical orbits of H.

By the theorem of Liouville [1, Proposition 3.3.4], the symplectic manifold (M,ω) is also a volume manifold, that is, the $2d$-form $\omega^d = \omega \wedge \overset{d}{\ldots} \wedge \omega$ is a volume form and induces a measure μ on M, which is called the Lebesgue measure associated to ω^d. Notice that the measure μ on M is invariant by the Hamiltonian flow.

Fixed a Hamiltonian $H \in C^2(M,\mathbb{R})$ any scalar $e \in H(M) \subset \mathbb{R}$ is called an *energy* of H and $H^{-1}(\{e\}) = \{p \in M : H(p) = e\}$ is the corresponding *energy level* set which is X_H^t-invariant. An *energy surface* $\mathcal{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$; we say that it is *regular* if it does not contain singularity points and in this case $\mathcal{E}_{H,e}$ is a regular compact $(2d-1)$-manifold. Moreover, H is constant on each connected component $\mathcal{E}_{H,e}$ of the boundary ∂M.

A *Hamiltonian system* is a triple $(H, e, \mathcal{E}_{H,e})$, where H is a Hamiltonian, e is an energy and $\mathcal{E}_{H,e}$ is a regular connected component of $H^{-1}(\{e\})$.

Due to the compactness of M, given a Hamiltonian function H and an H regular value $e \in H(M)$ the energy level $H^{-1}(\{e\})$ is the union of a finite number of disjoint compact connected components, separated by a positive distance. Given $e \in H(M)$, the pair $(H,e) \subset C^2(M,\mathbb{R}) \times \mathbb{R}$ is called a *Hamiltonian level*; if we fix $\mathcal{E}_{H,e}$ and a small neighborhood $\mathcal{W}$ of $\mathcal{E}_{H,e}$ there exist a small neighbourhood $\mathcal{U}$ of H and