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The inverse iteration method for Julia sets in the 3-dimensional space

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ABSTRACT

In this article, we introduce the adapted inverse iteration method to generate bicomplex Julia sets associated to the polynomial map $w^2 + c$. The result is based on a full characterization of bicomplex Julia sets as the boundary of a particular bicomplex cartesian set and the study of the fixed points of $w^2 + c$. The inverse iteration method is used in particular to generate and display in the usual 3-dimensional space bicomplex dendrites.

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1. Introduction

Fractal sets created by iterative processes have been greatly studied in the past decades (see [5,7,8,13]). After being displayed in the complex plane, they became part of the 3-dimensional space when Norton [14] gave straightforward algorithms using iteration with quaternions. The quaternionic Mandelbrot set defined by the quadratic polynomial of the form $q^2 + c$ was explored in [10,11]. However, as established in [1], it seems that no interesting dynamics could arise from this approach based on the local rotations of the classical sets. Another set of numbers revealed to be possibly more appropriate: Bicomplex Numbers. In [19], the author used bicomplex numbers to produce and display in 3D a Mandelbrot set for the quadratic polynomial of the form $w^2 + c$. Filled-in Julia sets were also generated using a method analogous to the classical one in the complex plane [19,20]. Since the bicomplex polynomial $P_c(w) = w^2 + c$ is the following mapping of $\mathbb{C}^2 : (z_1^2 - z_2^2 + c_1, 2z_1z_2 + c_2)$ where $w = z_1 + z_2\mathbf{i_2} := (z_1, z_2)$ and $c = c_1 + c_2\mathbf{i_2} := (c_1, c_2)$, bicomplex dynamics is a particular case of dynamics of several complex variables. More specifically, we note that this mapping is not a holomorphic automorphism of $\mathbb{C}^2$.

In this article, we study bicomplex Julia sets associated with the quadratic polynomial $w^2 + c$. We give a specific characterization of bicomplex Julia sets derived from a more general result in terms of the boundary of a bicomplex cartesian set. This characterization allows an easy display in the usual 3D space. The study of the inverse iterates and fixed points of $w^2 + c$ along with the characterization previously introduced lead to the first generalization of the inverse iteration method in two complex variables. This method, well known in the complex plane to generate Julia sets (see [4,15,17]), is used to generate and display in 3D a particular class of bicomplex Julia sets.

2. Preliminaries

2.1. Julia sets in the complex plane

Julia sets in the complex plane are defined according to the behavior of the forward iterates of a rational function. In this article, we restrict our study of Julia sets to a polynomial map that is easy to work with and has a dynamical system equivalent to the one of any polynomial map of degree two: $P_c(z) = z^2 + c$ where $z, c \in \mathbb{C}$ and c is fixed.

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First, we consider its iterates and fixed points. Next, we present some important and well known results about Julia sets.

The forward iterates of P_c are given by $P_c^0(z) = z$ and $P_c^n(z) = (P_c(z))^{\circ n} = (P_c \circ P_c^{(n-1)})(z)$ for $n \in \{1, 2, \ldots\}$. The inverse iterates are defined as $P_c^{-1}(z) := (P_c(z))^{\circ(-1)} = \{w \in \mathbb{C} | P_c(w) = z\}$ and $P_c^{-m}(w) := (P_c^m(w))^{\circ(-1)}$ for $m \in \{1, 2, \ldots\}$. The multivalued function $\sqrt{z-c}$ is associated to P_c^{-1}. The fixed points of P_c are found by solving the equation $P_c(z_0) = z_0$. A fixed point z_0 is said to be attractive if $0 \leqslant |2z_0| < 1$, repelling if $|2z_0| > 1$ and indifferent if $|2z_0| = 1$. For $c = \frac{1}{4}$, there is a single indifferent fixed point $z_0 = \frac{1}{2}$. Otherwise, there are two distinct fixed points and at least one of them is repelling (see [16]).

Let $\mathcal{K}_c = \{z \in \mathbb{C} | \{P_c^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$ be the filled-in Julia set associated to P_c. The Julia set related to P_c is denoted by $\mathcal{J}_c$ and defined as either one of the following:

1. The boundary of the filled-in Julia set: $\mathcal{J}_c = \partial\mathcal{K}_c$.
2. The set of points $z \in \mathbb{C}$ for which the forward iterates do not form a normal family at z (see [21] for details on normal families of functions).

The second definition leads to the following theorem that justifies the inverse iteration method. Note that it is stated for Julia sets $\mathcal{J}_P$ defined by any monic polynomial map P of degree $d \geqslant 2$. It is so valid for P_c. In [12], the classical statement of the result has been slightly modified from the one in [4].

Theorem 1. *Let P be a monic complex polynomial of degree $d \geqslant 2$.*

(i) *If $z_0 \in \mathcal{J}_P$ and V is any open neighborhood of z_0, then for any whole number $k_1 \geqslant 0$ there exists $N > k_1$ such that $\mathcal{J}_P \subseteq \bigcup_{k=k_1}^{N} P^k(V)$.*

(ii) *For any $z_1 \in \mathcal{J}_P$, the set of inverse iterates $\left\{\bigcup_{k=k_1}^{\infty} P^{-k}(z_1)\right\}$ is dense in $\mathcal{J}_P$ for all whole number $k_1 \geqslant 1$.*

To generate and display $\mathcal{J}_c$ in the complex plane, it suffices to take $z_1 \in \mathcal{J}_c$ and compute its inverse iteratively, up to a maximum number of iterations. Since the inverse is given by the complex square root function $\sqrt{z-c}$ which is multivalued, two different approaches may be used. The first one is to compute all branches of the inverse at every iteration, leading to a great number of points generated. The second option is to randomly choose one of the branches of the inverse at each iteration and compute only this one. This last approach seems more appropriate for it is faster and requires less memory space.

From [6], it is known that $\mathcal{J}_c$ is the closure of the set of repelling periodic points of P_c. Hence, for a starter z_1, one may choose a repelling fixed point of the polynomial map if $c \neq \frac{1}{4}$. If $c = \frac{1}{4}$, then $z_1 = \frac{1}{2}$ is a good starter for the algorithm since it is the only fixed point of P_c and known to be in $\mathcal{J}_c$ from [16]. For a good approximation of $\mathcal{J}_c$, a high enough number of iterations is needed.

Images from Fig. 1 are those of classical Julia sets produced by the inverse iteration method. For $c = \mathbf{i}, \mathcal{K}_\mathbf{c}$ is known to be a dendrite that is compact set, pathwise connected, locally connected, nowhere dense and that does not separate the plane [2]. A dendrite set is equal to its boundary and so $\mathcal{K}_c = \mathcal{J}_c$.

2.2. Bicomplex numbers

As presented in [18–20], bicomplex numbers arise from the work of Segre [22] and are defined as follows:

$$\mathbb{BC} = \{a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j} | a, b, c, d \in \mathbb{R}\}$$

where $\mathbf{i_1}^2 = \mathbf{i_2}^2 = -1, \mathbf{j}^2 = 1, \mathbf{i_1}\mathbf{i_2} = \mathbf{i_2}\mathbf{i_1} = \mathbf{j}, \mathbf{i_2}\mathbf{j} = \mathbf{j}\mathbf{i_2} = -\mathbf{i_1}$ and $\mathbf{i_1}\mathbf{j} = \mathbf{j}\mathbf{i_1} = -\mathbf{i_2}$. Since we can write $a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{j}$ as $(a + b\mathbf{i_1}) + (c + d\mathbf{i_1})\mathbf{i_2}$, the set of bicomplex numbers can be seen as

$$\mathbb{BC} = \{z_1 + z_2\mathbf{i_2} | z_1, z_2 \in \mathbb{C}(\mathbf{i_1})\}$$

where $\mathbb{C}(\mathbf{i_1})$ is the set of complex numbers with imaginary unit $\mathbf{i_1}$: $\mathbb{C}(\mathbf{i_1}) = \{x + y\mathbf{i_1} | x, y \in \mathbb{R} \text{ and } \mathbf{i_1}^2 = -1\}$. Hence, $\mathbb{BC}$ corresponds to a kind of complexification of the usual complex numbers. From [18], we can easily see that it is a commutative unitary ring. The set of bicomplex numbers is also sometimes denoted in the literature by $\mathbb{C}_2, \mathbb{T}, \mathbb{M}(2)$ or by the following complex Clifford algebras $\mathrm{Cl}_{\mathbb{C}}(1,0) \cong \mathrm{Cl}_{\mathbb{C}}(0,1)$.

An important property of bicomplex numbers is the unique representation using the idempotent elements $\mathbf{e_1} = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e_2} = \frac{1-\mathbf{j}}{2}$. In fact, $\forall w = z_1 + z_2\mathbf{i_2} \in \mathbb{BC}$, we have

$$\begin{aligned} z_1 + z_2\mathbf{i_2} &= (z_1 - z_2\mathbf{i_1})\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})\mathbf{e_2} \\ &= \mathcal{P}_1(w)\mathbf{e_1} + \mathcal{P}_2(w)\mathbf{e_2} \end{aligned}$$

where the projections $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{BC} \longrightarrow \mathbb{C}(\mathbf{i_1})$ are defined as $\mathcal{P}_1(z_1 + z_2\mathbf{i_2}) := z_1 - z_2\mathbf{i_1}$ and $\mathcal{P}_2(z_1 + z_2\mathbf{i_2}) := z_1 + z_2\mathbf{i_1}$.

The usual operations of addition and multiplication can be done term-by-term using this representation:

(i) $(z_1 + z_2\mathbf{i_2}) + (s_1 + s_2\mathbf{i_2}) = [(z_1 - z_2\mathbf{i_1}) + (s_1 - s_2\mathbf{i_1})]\mathbf{e_1} + [(z_1 + z_2\mathbf{i_1}) + (s_1 + s_2\mathbf{i_1})]\mathbf{e_2}$.

(ii) $(z_1 + z_2\mathbf{i_2}) \cdot (s_1 + s_2\mathbf{i_2}) = [(z_1 - z_2\mathbf{i_1})(s_1 - s_2\mathbf{i_1})]\mathbf{e_1} + [(z_1 + z_2\mathbf{i_1})(s_1 + s_2\mathbf{i_1})]\mathbf{e_2}$.

(iii) $(z_1 + z_2\mathbf{i_2})^n = (z_1 - z_2\mathbf{i_1})^n\mathbf{e_1} + (z_1 + z_2\mathbf{i_1})^n\mathbf{e_2}$ for $n = 0, 1, 2, \ldots$.

The real modulus of $w = z_1 + z_2\mathbf{i_2} \in \mathbb{BC}$ is given by $||w|| = \sqrt{|z_1|^2 + |z_2|^2}$ where $|\cdot|$ is the Euclidian norm in $\mathbb{C}(\mathbf{i_1})$. Writing $z_1 = a + b\mathbf{i_1}$ and $z_2 = c + d\mathbf{i_1}$, we have $||w|| = \sqrt{a^2 + b^2 + c^2 + d^2}$ which is the Euclidian norm in $\mathbb{R}^4$.

The square root of a bicomplex number $w = z_1 + z_2\mathbf{i_2}$ is given in terms of the complex square roots of its idempotent components:

$$\begin{aligned} \sqrt{z_1 + z_2\mathbf{i_2}} &= \sqrt{z_1 - z_2\mathbf{i_1}}\,\mathbf{e_1} + \sqrt{z_1 + z_2\mathbf{i_1}}\,\mathbf{e_2} \\ &= \sqrt{\mathcal{P}_1(w)}\,\mathbf{e_1} + \sqrt{\mathcal{P}_2(w)}\,\mathbf{e_2}. \end{aligned}$$

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