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A class of viscous p-Laplace equation with nonlinear sources ☆

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ABSTRACT

In this paper, we prove the global existence of solutions to the initial boundary value problem of a viscous p-Laplace equation with nonlinear sources. The asymptotic behavior of solutions as the viscous coefficient k tends to zero is also investigated. In particular, we discuss the H^1-Galerkin finite element method for our problem and establish the error estimates for two semi-discrete approximate schemes.

1. Introduction

This paper concerns the following initial boundary value problem of the viscous p-Laplace equation with sources in one spatial dimension

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D(|Du|^{p-2}Du) + u^q, \quad (x,t) \in Q_T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.3}$$

where $Q_T \equiv (0,1) \times (0,T)$, $T > 0$ is a given constant, $p \geqslant 2$, $0 < q < p-1$, $D = \partial/\partial x$, $k > 0$ denotes the viscous coefficient, $u_0 \in C^{2+\alpha}[0,1]$ for some $\alpha \in (0,1)$ with $u_0(0) = u_0(1) = 0$.

Equations of type (1.1) with one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise from the study of flows of fluids through fissured rock [1], thermodynamics [2], the unidirectional propagation of nonlinear, dispersive, long waves [3], the aggregation of populations [4], etc. One area in which the problem we consider arises from study of shearing flows of incompressible simple fluids. The quantity $|Du|^{p-2}Du + k\frac{\partial Du}{\partial t}$ can be viewed as an approximation to the stress functional during such a flow, and the term $\frac{\partial Du}{\partial t}$ can be interpreted as viscous relaxation effects, or viscosity, see [5]. Besides, when the influence of many factors, such as the molecular and ion effects are considered, the nonlinear term $D(|Du|^{p-2}Du)$ appears to be instead of D^2u in the initial linear pseudo-parabolic model

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u,$$

which was fully studied in [6].

Mathematical study of problems similar to (1.1)–(1.3) goes back to works of Davis [7] in 1972, who proposed the initial-boundary value problem of the following nonlinear Sobolev–Galpern equation

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D\sigma(Du), \quad (x,t) \in Q_T,$$

which was also considered by [8–10]. He showed the existence of a unique classical solution (under the conditions $u_0 \in C^2(0,1)$, $\sigma \in C^2(-\infty,\infty)$, $\sigma(0) = 0$ and $\sigma'(\xi) > 0$), and the convergence of these solutions to the unique weak solution of the quasilinear parabolic problem with $k = 0$, provided that $\sigma'(\xi)$ has a positive lower bound. Subsequently, nonlinear pseudo-parabolic equations, even including degeneration or singularity have been investigated by many authors. For example, see the seminal contribution [11] for well-posedness of abstract model using the theory of semigroups of nonlinear contractions by hyper-accretive relations in Banach space; [12] for quasilinear pseudo-parabolic equations with degeneration in the time derivative

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term and including memory terms; [13] for local in time existence of pseudo-parabolic equation modeling solvent uptake in polymeric solids, supposing nonnegative compactly supported initial datum; global existence results of degenerate pseudo-parabolic equations describing sequestration of the carbon dioxide in unminable coal seams and unsaturated flow in porous media with dynamic capillary pressure can be found in the papers [14,15].

The study on asymptotic behavior of solutions when the third order term vanishing is also very important. The investigation of the capillarity limit in [16] mentions that it leads to the Buckley–Leverett equation with discontinuous solutions which do not satisfy Oleinik's entropy condition. Authors of [17,18] also discuss the existence of traveling wave solutions in one spatial dimension and their relation to non-standard shock solutions to hyperbolic conservation laws via the viscos term tending to zero.

Regarding numerical analysis, we mention pioneering results in [19] (super-convergence of a finite element approximation), [20,21] (time-stepping Galerkin methods both for nonlinear pseudo-parabolic problems), or more recently [22] (discuss the case of discontinuous initial data), [23] (where existence, uniqueness and error estimates are obtained for a similar kind of problems, but with a different kind of nonlinear second order operator). For a broad overview of the various numerical techniques and results for such problems, we refer to [24].

Recently, more considerable attentions have been paid to the study of Sobolev equations with nonlinear sources. When $p = 2$, [25,26] investigated the critical Fujita exponent and large time behavior for the case $q > 0$. When $p > 2$, [27] studied the questions of global solvability and blow up in finite time for the case $q = p + 1$. Apparently, the case $p > 2$ has not been fully studied in mathematical terms, and here we focus on the case $p > 2$ with $0 < q < p - 1$.

In this paper, based on a priori estimates we prove the global existence of solutions to (1.1)–(1.3). Due to the degeneracy of the equation, the problem (1.1)–(1.3) has no classical solutions in general and thus we consider its weak solutions in the following sense.

Definition 1.1. A function u is said to be a solution of the initial-boundary problem (1.1)–(1.3), if $u \in C^{1+1/2,\mu}(\overline{Q}_T) \cap H^{2,1}(Q_T)$ with $\mu \in (0,1/2)$, $D^2u_t \in L^2(Q_T)$, u satisfies (1.2) and (1.3) and the following equality

$$\int\int_{Q_T} \frac{\partial u}{\partial t}\varphi dxdt - k\int\int_{Q_T} \frac{\partial D^2 u}{\partial t}\varphi dxdt + \int\int_{Q_T} (|Du|^{p-2}Du)D\varphi dxdt - \int\int_{Q_T} u^q\varphi dxdt = 0$$

for any $\varphi \in C^1(\overline{Q}_T)$ with $\varphi(0,t) = \varphi(1,t) = 0$ for all $t \in (0,T)$.

Besides, we give the asymptotic behavior of these solutions as k tends to zero, namely the weak solutions of (1.1)–(1.3) tend to the weak solution of the corresponding parabolic equation. In particular, we consider the H^1-Galerkin finite element method for our problem and derive error estimates for two semi-discrete approximate schemes.

This paper is organized as follows. Section 2 provides the global existence of solutions to the problem (1.1)–(1.3). In Section 3, the asymptotic behavior of solutions as the viscous coefficient k tends to zero is obtained. Finally, two weak formulations of a standard H^1-Galerkin finite element and an H^1-Galerkin mixed finite element are given in Section 4, respectively, and error estimates are given for semi-discrete schemes for our problem (1.1)–(1.3).

2. Existence

In this section, we obtain the following existence result

Theorem 2.1. *Assume that $u_0 \in C^{2+\alpha}[0,1]$ with $\alpha \in (0,1)$. Then the problem (1.1)–(1.3) admits at least one solution $u \in C^{1+\mu_1,\mu_2}(\overline{Q}_T) \cap H^{2,1}(Q_T)$ with $\mu_1 \in (0,1/2)$ and $\mu_2 \in (0,1/2)$, $D^2u_t \in L^2(Q_T)$.*

To discuss the solvability of the problem (1.1)–(1.3), we consider the following regularized equation

$$\frac{\partial u_\varepsilon}{\partial t} - k\frac{\partial D^2 u_\varepsilon}{\partial t} = D\left((|Du_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} Du_\varepsilon\right) + u_\varepsilon^q, \quad (x,t) \in Q_T, \tag{2.1}$$

where ε is a positive constant. Firstly, we study the existence of classical solutions to the problem (2.1), (1.2) and (1.3).

Proposition 2.1. *Assume that $u_0 \in C^{2+\alpha}[0,1]$ with $\alpha \in (0,1)$. The problem (2.1), (1.2) and (1.3) admits at least one classical solution $u_\varepsilon \in C^{2+\gamma,1}(\overline{Q}_T)$ with $D^2u_{\varepsilon t} \in C^{\gamma,0}(\overline{Q}_T)$, $\gamma = \min\{1/2,\alpha\}$.*

We will prove this proposition by applying the following Leray–Schauder's fixed point theorem.

Leray–Schauder's Fixed Point Theorem. Let X be a Banach space, $F(u,\sigma)$ be a mapping from $X \times [0,1]$ to X satisfying the following conditions:

(i) F is a compact mapping;
(ii) $F(u,0) = 0$, $\forall u \in X$;
(iii) There exists a constant $M > 0$, such that for any $u \in X$, if $u = F(u,\sigma)$ holds for some $\sigma \in [0,1]$, then $\|u\|_X \leqslant M$.

Then the mapping $F(\cdot,1)$ has a fixed point, that is, there exists $u \in X$, such that $u = F(u,1)$.

In terms of the above theorem, we can study the problem (2.1), (1.2) and (1.3) by considering the following equation

$$\frac{\partial u_\varepsilon}{\partial t} - k\frac{\partial D^2 u_\varepsilon}{\partial t} = l(x,t)D^2u_\varepsilon + \sigma n(x,t), \quad (x,t) \in Q_T$$

subject to the following initial and boundary value conditions

$$u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, \quad t \in [0,T], \tag{2.2}$$

$$u_\varepsilon(x,0) = \sigma u_0(x), \quad x \in [0,1], \tag{2.3}$$

where $l(x,t) = (|Dv|^2 + \varepsilon)^{\frac{p-2}{2}} + (p-2)(|Dv|^2 + \varepsilon)^{\frac{p-2}{2}-1}|v|^2$, $n(x,t) = v^q(x,t)$ with $v \in C^{1+1/2,\mu}(\overline{Q}_T)$, $\mu \in (0,1/2)$, $v(0,t) = v(1,t) = 0$ for $t \in [0,T]$. Here, σ is a parameter taking value on the interval $[0,1]$. By the result in [28], we know that the above problem admits a unique solution $u_\varepsilon \in C^{2+\gamma,1}(\overline{Q}_T)$ with $D^2u_{\varepsilon t} \in C^{\gamma,0}(\overline{Q}_T)$, where $\gamma = \min\{1/2,\alpha\}$. Hence we define a mapping F as follows