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Intra-orbit separation of dense orbits of dendrite maps ☆

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ABSTRACT

In this note, we will generalize the notion of separation index, which was introduced by Manjunath et al. (2006), to dendrite maps, and use the notion to characterize the intra-orbit separation for the orbits of continuous transitive dendrite maps. We will show: (i) For a dendrite map f, the separation index γ is greater than zero if and only if the set of fixed points of f is not an arc connected subspace. (ii) If the separation index γ of a dendrite map f is greater than zero, then for every $0 < \tau < \gamma$ and any pair of distinct points x and y on a dense orbit, $\{x, y\}$ is a Li–Yorke pair of modulus τ.

1. Introduction

Let (X, d) be a metric space. For any $A \subset X$, denote by diam$A$ the diameter of A. For any $y \in X$ and any $r > 0$, write $B(y, r) = \{x \in X : d(y, x) < r\}$. Let $\mathbf{N}$ be the set of all positive integers and $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

Denote by $C^0(X)$ the set of all continuous maps from X to X. For any $f \in C^0(X)$, let f^0 be the identity map of X and $f^n = f \circ f^{n-1}$ be the composition map of f and f^{n-1}. A point $x \in X$ is called a periodic point of f with period n (briefly called a n-periodic point of f) if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leqslant i < n$. The orbit of x under f is the set $O(x, f) \equiv \{f^n(x) : n \in \mathbf{Z}_+\}$. A 1-periodic point is generally called a fixed point. The set of fixed points, n-periodic points and periodic points of f is denoted by $F(f)$, $P_n(f)$ and $P(f)$ respectively.

Let $f \in C^0(X)$. If for every nonempty open subsets U and V of X, there exists $n \in \mathbf{N}$ such that $f^n(U) \cap V \neq \emptyset$, then the map f is said to be topologically transitive (or just transitive) and the system (X, f) is said to be topologically transitive. The definition of transitivity for compact metric spaces is equivalent to the statement that X has a dense orbit under f (see [5,14]). Any point with dense orbit is called a transitive point. Given a positive real number τ, a pair of points $\{x, y\}$ is called a Li–Yorke pair with modulus τ (or simply called a τ-scrambled pair) if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \tau$$

and

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

A set is said to be a scrambled set if any two distinct points of the set form a Li–Yorke pair with some positive modulus.

In [10], Manjunath et al. introduced the notion of intra-orbit separation for the orbits of continuous transitive maps on a compact interval to demonstrate separation of two points on a given dense orbit. They associated a non-negative real number γ with a transitive interval map f called the separation index of the map f. For a transitive interval map f having at least two fixed points, they showed: (i) the separation index γ is positive, (ii) for every $0 < \tau < \gamma$ and any pair of distinct points x and y on a dense orbit, $\{x, y\}$ is a Li–Yorke pair of modulus τ.

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In recent years, dynamical systems on dendrites have been studied by many authors. The subjects such as the topological structure of minimal sets, the depth of the center, the topological entropy of dendrite maps are deeply discussed (see [1–3,6–8,12]). The interest in these subjects is motivated in part by the fact that dendrites appear as Julia sets in complex dynamics (see [4] or [13]). In [8], Kato constructed a dendrite D and a continuous map $f \in C^0(D)$ such that $\overline{R(f)} \neq \overline{P(f)}$, where $R(f)$ is the set of recurrent points of f. However, Mai and Shi [9] proved that if T is a dendrite of which the cardinal number of endpoints is less than c (the cardinal number of the continuum), then $\overline{R(f)} = \overline{P(f)}$ for any $f \in C^0(T)$.

A space homeomorphic to $[0,1]$ (resp. to the unit circle S_1 in the complex plane $\mathbf{C}$) is called an arc (resp. a circle). A compact, connected and locally connected metric space is called a Peano continuum. A Peano continuum which contains no circle is called a dendrite. There are many known properties of Peano continua and of dendrites (see [11]). Let (T,d) be a nondegenerate dendrite. Every continuous map $f \in C^0(T)$ is called a dendrite map. For any $x \in T$, denote by val(x) = val(x,T), called the valence of x in T, the cardinal number of the family of connected components of $T - \{x\}$. The point x is called an endpoint if val(x) = 1. Denote by $E(T)$ the set of all endpoints of T. For any arc A, we also write ∂A for $E(A)$. For any two different points $x,y \in T$, it is well known that there is a unique arc in T, denoted by $[x,y]$ or $[x,y]_T$, such that $\partial[x,y] = \{x,y\}$. Write $[x,y) = (y,x] = [x,y] - \{y\}$, and $(x,y) = [x,y) - \{x\}$. Denote by $T_x(y)$ the connected component of $T - \{x\}$ containing y. In addition, we write $[x,x] = x$.

For any an arc $[x,y]$ in T, write $R_{xy} : T \longrightarrow [x,y]$ be the retraction (called the first point map of T for $[x,y]$, see [11, P176]) such that $[z, R_{xy}(z)] \cap [x,y] = R_{xy}(z)$ for any $z \in T$.

In this note, we will generalize the notion of intra-orbit separation for the orbits of transitive interval maps in [10] to dendrite maps.

2. Intra-orbit separation for dendrite maps

To characterize the intra-orbit separation for a dendrite map, we first introduce the following notion. Let T be a dendrite and $f \in C^0(T)$, define a subset $S(f^n)$ of T for each $n \in \mathbf{N}$ as follows:

$$S(f^n) = \{x \in T : f^n(x) \neq x, \text{there is } y \in T \text{ such that } [y,x] \subset [f^n(y), f^n(x)) \text{ and } x \in (y, f^n(x))\}.$$

For any $x \in S(f^n)$, write

$$S(x,f^n) = \{y \in T : [y,x] \subset [f^n(y), f^n(x)) \text{ and } x \in (y, f^n(x))\}.$$

From the compactness and the local connectivity it is easy to show the following lemma.

Lemma 2.1 (*see [9]*). *Let (T,d) be a dendrite. Then, for every $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon) > 0$ such that, for any $x,y \in T$ with $d(x,y) \leqslant \mu$, diam$[x,y] < \varepsilon$.*

The following Lemma 2.2 *and* Lemma 2.3 *are from* [9].

Lemma 2.2 (*see [9]*). *Let $[x,y]$ be an arc in a dendrite (T,d), and $w \in [x,y)$. Let $\varepsilon = d(w,y)$ and $\mu = \mu(\varepsilon)$ be the same as in* Lemma 2.1. *Then $[x,v] \supset [x,w]$ for any $v \in T$ with $d(v,y) \leqslant \mu$.*

Lemma 2.3 (*see [9]*). *Let $f \in C^0(T)$ and $x \in T$ with $x \neq f(x)$. Then $T_x(f(x)) \cap F(f) \neq \emptyset$.*

Lemma 2.4. *Let $f \in C^0(T)$. Then $S(x,f^n) \cap F(f^n) \neq \emptyset$ for any $n \in \mathbf{N}$.*

Proof. Let $y \in S(x,f^n)$. Then $[y,x] \subset [f^n(y), f^n(x))$ and $x \in (y, f^n(x))$. Since $f^n([y,x)) \supset [y,x]$, there exists a sequence of points $z_1 = y, \ldots, z_m, \ldots \in [y,x)$ satisfying $z_j \in [y, z_{j+1}]$ and $f^n(z_{j+1}) = z_j$ for each $j \geqslant 1$. Let $\lim_{m\to\infty} z_m = p$. It is obvious that $f^n(p) = p \in [y,x)$, which implies $p \in S(x,f^n) \cap F(f^n)$. □

Remark 2.5. It is obvious that $S(f^n) \cap E(T) = \emptyset$ and $S(x,f^n) \setminus F(f^n) \subset S(f^n)$ for any $n \in \mathbf{N}$.

Proposition 2.6. *Let $f \in C^0(T)$ and $S(f) \neq \emptyset$. Then $S(x,f)$ is a closed subset of T for any $x \in S(f)$.*

Proof. Let $x \in S(f)$ and a sequence of points $y_1, y_2, \ldots \in S(x,f)$ satisfying $\lim_{n\to\infty} y_n = y$. it is easy to show $y \neq x$ (otherwise, if $y = x$, then there is a large n such that $[y_n, x] \cap [f(y_n), f(x)) = \emptyset$, which is a contradiction).

We claim $R_{xf(x)}(y) = x$. Indeed, if $R_{xf(x)}(y) \neq x$, then

$$\begin{aligned}\text{diam}[y_n,y] &= \text{diam}\{[y_n,x] \cup [x, R_{xf(x)}(y)] \cup [R_{xf(x)}(y), y]\}\\ &\geqslant \text{diam}\ [x, R_{xf(x)}(y)],\end{aligned}$$

which with Lemma 2.1 implies $\lim_{n\to\infty} y_n \neq y$, a contradiction. In a similar fashion, we may show $R_{xf(x)}(f(y)) = x$.

Now we show $R_{yx}(f(y)) = y$. Indeed, if $R_{yx}(f(y)) \neq y$, then according to Lemma 2.2 and continuity of f we may suppose that:

(1) There is $u \in (y, R_{yx}(f(y)))$ such that $[y, f(y_n)] \supset [y,u]$ for any $n \in \mathbf{N}$.
(2) $R_{yx}(y_n) \in [y,u]$ for any $n \in \mathbf{N}$. Thus

$$\begin{aligned}\text{diam}[f(y_n), f(y)] &= \text{diam}\{[f(y_n), y_n] \cup [y_n, R_{yx}(y_n)]\\ &\quad \cup [R_{yx}(y_n), R_{yx}(f(y))] \cup [R_{yx}(f(y)), f(y)]\}\\ &\geqslant \text{diam}[R_{yx}(y_n), R_{yx}(f(y))].\end{aligned}$$

Since

$$\begin{aligned}\text{diam}[y_n,y] &= \text{diam}\{[y_n, R_{yx}(y_n)] \cup [R_{yx}(y_n), y]\}\\ &\geqslant \text{diam}[R_{yx}(y_n), y]\end{aligned}$$

and $\lim_{n\to\infty} y_n = y$, we have $\lim_{n\to\infty} R_{yx}(y_n) = y$, which with Lemma 2.1 implies $\lim_{n\to\infty} f(y_n) \neq f(y)$, a contradiction.

Therefore $y \in S(x,f)$ and $S(x,f)$ is a closed subset of T. □