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Non-isospectral flows of noncommutative differential-difference KP equation

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ABSTRACT

We present master symmetries of noncommutative differential-difference KP equation by considering Sato approach, where the field variables are defined over associative algebras. The Lie algebraic structures of generalized and master symmetries are given. They form a Virasoro Lie algebraic structure.

1. Introduction

Integrable systems are known for its rich and diverse mathematical structures and applications. In the past four decades, the research in searching soliton systems and investigating their integrability properties have made significant progress [1]. Date et al. systematically derived soliton system and their Lax pairs for both continuous and discrete equations by using group theoretic approach in a series of papers [2–6]. It is well-known that Sato theory is a powerful mathematical approach to derive Lax pairs, soliton solutions, conservation laws etc. [7] for KP hierarchy. This approach has further extended to differential difference set up by [8–11] and obtained differential-difference KP equation, generalized symmetries and conservation laws. Zhang et al. continued this analysis and found the conservation laws and solutions of differential-difference KP in a much simpler way [12,13]. Searching for various symmetries, master symmetries and Lie algebraic structures for continuous and lattice equations is an important topic for many years [15–21]. They always play a fundamental role in integrable systems. In [19], Ma proposed a method to construct Lax representation for isospectral and non-isospectral hierarchies of $(1+1)$ dimensional evolution equations.Further developments have been made in [16,20,21] for lattice equations as well. Moreover, using this one can study the underlying infinite dimensional Lie algebraic structures of the concerned system. Recently, many studies have been carried out in finding the symmetries and their Lie algebraic structures for differential-difference systems [8,10,12–14]. The non-isospectral flows assume its importance in their own right. It requires a separate study to find out various properties of them concerning the integrability. It is interesting to note that solutions of certain non-isospectral equations were carried out by Zhang et al. [22–24]. In recent years, noncommutative version of soliton equations attracted lot of attention in the area of integrable nonlinear partial differential equations [25–43]. They arise in many situations, where the field variables, for example, might be square matrices or quarternions etc. There is an another way in defining these equations through $*$ or Moyal product [36–41]. In this paper, the non-commutative version is obtained by assuming that the coefficients in the pseudo-difference operator do not commute with each other. In the literature not enough results are available for non-commutative version of soliton equations [28,29,38,41,40]. Hence, there are huge possibilities exist to explore in this domain and obtain interesting results. Olver and Sokolov [26] developed a systematic procedure to find generalized symmetries, recursion operators and Hamiltonian structures of certain $(1+1)$ dimensional

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noncommutative evolution equations. In a series of papers, Hamanaka extended Sato theory for noncommutative settings and obtained continuous noncommutative KP (NCKP) equation [36–38] and its various reductions. A detailed investigation about extended KP and its noncommutative version have been discussed in [43,44]. However, not enough is known about various integrablity properties of noncommutative differential-difference soliton equations. Quasi-determinant [31] solutions of non-abelian Hirota-Miwa equation has been obtained by Nimmo et al. [32] and Darboux transformation and many further results have been obtained in [34,35,42] for number of noncommutative differential- difference equations. From the literatures cited above, it is clear that investigating algebraic structures of symmetries for noncommutative soliton systems is crucial to understand the integrability of these class of equations. In [45] some of the present authors, derived non-isospectral flows of noncommutative KP and obtained their algebraic structures. In this paper, we derive iso and non-isospectral flows/master symmetries and Lie algebras of noncommutative differential-difference KP equations through Sato approach and prove that Lie algebraic structure is indeed Virasoro type as in the commutative case.

2. Isospectral and non-isospectral flows of NCDΔKP equation

Let us consider the difference analogue of a quasi-differential operator

$$L = \Delta + u_0 + u_1\Delta^{-1} + \cdots + u_j\Delta^{-j} + \cdots, \tag{2.1}$$

where $u_s \triangleq u_s(n,t) = u_s(n,y,t_1,t_2,\ldots)$, $s = 0,1,2,\ldots$, $t = (t_1,t_2,\ldots)$ are the noncommutative field variables, i.e. the field variables over associative algebra $\mathcal{A}$ and Δ denotes the forward difference operator defined by $\Delta f(n) = (E-1)f(n) = f(n+1) - f(n)$. The operators Δ and E are connected by $\Delta \equiv E - 1$ and $\Delta\,\Delta^{-1} = \Delta^{-1}\Delta = 1$.

We recall the following properties for difference operator Δ,

$$\begin{aligned}
&\Delta(XY) = (EX)(\Delta Y) + (\Delta X)Y\\
&\Delta^2 X = E^2X - 2EX + X,\\
&\Delta^{-1}(XY) = E^{-1}X\Delta^{-1}Y - \Delta^{-1}(\Delta(E^{-1}X)\Delta^{-1}Y), \quad \forall\, X,Y \in \mathcal{A}\\
&\Delta^{-1}(y_n) = \sum_1^{n-1} y_n + \text{constant independent of } n,\\
&\Delta^{-1}(1) = n + \text{constant independent of } n.
\end{aligned}$$

2.1. Isospectral NCDΔKP

For constructing the iso and non-isospectral flows of KP equation the Lax triad approach is more convenient to use rather than Lax Pair method. Here, we use the same to obtain the associated flows of NCDΔKP equation. Throughout this paper, we follow Lax triad approach. First, we consider the spectral problem corresponds to NCDΔKP with eigenvalue η and eigenfunction ϕ as follows:

$$L\phi = \eta\phi, \tag{2.2a}$$
$$\phi_y = A_1\phi, \tag{2.2b}$$
$$\phi_{t_s} = A_s\phi, \tag{2.2c}$$

where $\eta_{t_s} = \frac{\partial\eta}{\partial t_s} = 0$ and $A_s = (L^s)_+$, projection into the positive powers of Δ, satisfying the boundary condition

$$A_s|_{u=0} = \Delta^s. \tag{2.3}$$

Then the compatibility conditions of the linear problems (2.2) lead to

$$L_y = [A_1, L], \tag{2.4a}$$
$$L_{t_s} = [A_s, L], \tag{2.4b}$$
$$A_{1,t_s} = A_{s,y} - [A_1, A_s], \tag{2.4c}$$

From the fact, $A_s = (L^s)_+$, all the A_s can be determined and the first few explicit forms of A_s are given below:

$$A_1 = \Delta + u_0, \tag{2.5a}$$
$$A_2 = \Delta^2 + (\Delta u_0 + 2u_0)\Delta + (\Delta u_0 + u_0^2 + \Delta u_1 + 2u_1), \tag{2.5b}$$
$$A_3 = \Delta^3 + a_1\Delta^2 + a_2\Delta + a_3, \tag{2.5c}$$

with

$$a_1 = \Delta^2 u_0 + 3\Delta u_0 + 3u_0,$$

$$\begin{aligned}
a_2 = {}& 2\Delta^2 u_0 + 3\Delta u_0 + 3u_0^2 + 2u_0\Delta u_0 + \Delta u_0 u_0 + (\Delta u_0)^2\\
&+ 3u_1 + 3\Delta u_1 + \Delta^2 u_1,
\end{aligned}$$

$$\begin{aligned}
a_3 = {}& \Delta^2 u_0 + 3u_0u_1 + 2u_1u_0 + 2u_0\Delta u_0 + (\Delta u_0)u_0 + (\Delta u_0)^2\\
&+ (\Delta u_0)\Delta u_1 + 2u_0\Delta u_1 + (\Delta u_1)u_0 + (\Delta u_0)u_1 + u_1E^{-1}u_0\\
&+ 2\Delta^2 u_1 + 3\Delta u_1 + 3\Delta u_2 + 3u_2 + \Delta^2 u_2 + u_0^3.
\end{aligned}$$

Using (2.5a) in (2.4a), we get

$$u_{0,y} = q_{10} = \Delta u_1, \tag{2.7a}$$

$$u_{1,y} = q_{11} = \Delta u_1 + \Delta u_2 + u_0u_1 - u_1E^{-1}u_0, \tag{2.7b}$$

$$\begin{aligned}
u_{2,y} = q_{12} = {}& \Delta u_3 + \Delta u_2 + u_0u_2 + u_1E^{-1}u_0 - u_2E^{-2}u_0\\
&- u_1E^{-2}u_0, \ldots
\end{aligned} \tag{2.7c}$$

From (2.7), we can easily express $u_j (j > 0)$ through u_0, i.e.,

$$u_1 = \Delta^{-1}u_{0,y}, \tag{2.8a}$$

$$\begin{aligned}
u_2 = {}& \Delta^{-2}u_{0,yy} - \Delta^{-1}u_{0,y} - \Delta^{-1}(u_0\Delta^{-1}u_{0,y})\\
&+ \Delta^{-1}(\Delta^{-1}u_{0,y}E^{-1}u_0),\\
&\ldots
\end{aligned} \tag{2.8b}$$

By employing (2.4c), one can have the evolution equations for u_0:

$$u_{0,t_1} = q_{10} = u_{0,y}, \tag{2.9a}$$

$$\begin{aligned}
u_{0,t_2} &= q_{20}\\
&= \Delta u_{0,y} + u_{0,y}u_0 + u_0u_{0,y} + \Delta u_{1,y} + 2u_{1,y}\\
&\quad + [\Delta u_1, u_0] + 2[u_1, u_0] - \Delta^2 u_1 - 2\Delta u_1,
\end{aligned} \tag{2.9b}$$