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Homoclinic orbits for second-order Hamiltonian systems with subquadratic potentials ☆

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ABSTRACT

In this paper we consider a class of subquadratic second-order Hamiltonian systems and new results about the existence and multiplicity of homoclinic orbits are obtained by using the Minimizing Theorem and the Clark's Theorem respectively and a new compact imbedding theorem is also proved.

1. Introduction and main result

Consider the following second order nonautonomous Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0 \tag{1}$$

where $L \in C(R,R^N)$ is a symmetric matrix valued function, $W \in C^1(R \times R^N,R)$. We say that a nonzero solution u of problem (1) is homoclinic (to 0) if $u(t) \to 0$ as $|t| \to \infty$.

In the last two decades, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been intensively studied by many mathematicians. Indeed the existence of homoclinic orbits for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been recognized from Poincaré [1]. If $L(t)$ and $W(t,x)$ are independent of t or periodic in t, many authors have studied the existence of homoclinic orbits for Hamiltonian systems, see for instance [2–9] and a more general case is considered in recent papers [10–13]. In this case, the existence of homoclinic orbits is obtained by going to the limit of periodic solutions of approximating problems. In recent years, Concentration Compactness Principle has also been widely used to deal with the perturbation of periodic or autonomous problems, for example [14,15].

If $L(t)$ and $W(t,x)$ are neither autonomous nor periodic, the problem is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding. Rabinowitz and Tanaka [16] study without any periodicity assumption and obtain the existence of homoclinic orbits of problem (1) by using a variant of the Mountain Pass Theorem without the Palais–Smale condition, under the following condition.

(L) $L \in C(R, R^{N^2})$ is a symmetric and positively definite matrix for all $t \in R$ and there exists a continuous function $l:R \to R$ such that $l(t) > 0$ for all $t \in R$ and

$$(L(t)x, x) \geqslant l(t)|x|^2, \; l(t) \to \infty \text{ as } |t| \to \infty.$$

Assuming coercivity assumption (L), Omana and Willem [17] obtain an improvement on the latter result by employing a new compact embedding theorem, in fact, they show that the (PS) condition is satisfied and obtain the existence and multiplicity of homoclinc orbits of problem (1) by using the usual Mountain Pass Theorem. Under condition (L) some other cases are considered in recent papers, for example [18–21]. However, it is frequent occurrence that the global positive definiteness of $L(t)$ is not satisfied. In [22], defining the smallest eigenvalue of $L(t)$ as following

$$l(t) = \inf_{|x|=1} (L(t)x, x),$$

and assuming $l(t)$ satisfies

(L_ξ) there exists a constant $\xi \leqslant 2$ such that

$$l(t)|t|^{\xi-2} \to +\infty$$

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as $|t| \to \infty$, the author investigates the existence and multiplicity of homoclinic orbits of problem (1) for the case that $L(t)$ is unnecessary uniformly positively definite for all $t \in R$, which has been complemented by [23–28]. Korman and Lazer [29] remove the technical coercivity in the case that $L(t)$ and $W(t,x)$ are even in t, by approximating homoclinic orbits from solutions of boundary value problems, which is complemented by Lv and Tang [30]. Recently, Tang and Lin [31], Yuan and Zhang [32] obtain the existence of homoclinic orbits of problem (1) under the condition that L is uniformly definite and bounded from below without the coercivity and even assumption.

Most of papers above treat the superquadratic case (see [2–18,21–23,25,27,28,32]) and some papers treat the subquadratic case (see[18–22,25,28,31]). In this paper, we will consider the existence and multiplicity of homoclinic orbits for subquadratic second-order Hamiltonian systems. Here, we list the respect results in [19,20,31] specifically.

Theorem A [19, Theorem 1.1]. *Assume that L satisfies (L) and W satisfies(H_1) $W(t,x) = a(t)|x|^r$, $a:R \to R^+$ is a positive continuous function such that*

$$a \in L^2(R,R) \cap L^{\frac{2}{2-r}}(R,R)$$

and $1 < r < 2$ is a constant.

Then problem (1) possesses a nontrivial homoclinic orbit.

Theorem B [20, Theorem 1.2]. *Assume that L satisfies (L) and W satisfies(H_2) $W(t,x) = a(t)|x|^r$ where $a:R \to R^+$ is a continuous function such that*

$$a \in L^{\frac{2}{2-r}}(R,R)$$

and $1 < r < 2$ is a constant.

Then problem (1) possesses infinitely many homoclinic orbits. But in fact in Theorem B the condition respect to a is not sufficient and the condition that a is positive is used in the proof of the Lemma 3.1 in [20].

Theorem C [31, Theorem 1.1]. *Assume that L satisfies(L') $L \in C(R,R^{N\times N})$ is definite symmetric matrix for all $t \in R$ and there exists a constant $\beta > 0$ such that*

$$(L(t)x,x) \geqslant \beta|x|^2$$

for all $(t,x) \in R \times R^N$;
and W satisfies(H_3) There exist two constants $1 < r_1 < r_2 < 2$ and two functions $a_1,\ a_2 \in L^{\frac{2}{2-r_1}}(R,[0,+\infty))$ such that

$$|W(t,x)| \leqslant a_1(t)|x|^{r_1}$$

for all $(t,x) \in R \times R^N, |x| \leqslant 1$, and

$$|W(t,x)| \leqslant a_2(t)|x|^{r_2}$$

for all $(t,x) \in R \times R^N, |x| \geqslant 1$;
(H_4) There exist two functions $b \in L^{2/(2-r_1)}(R,[0,+\infty))$ and $\varphi \in C([0,+\infty),[0,+\infty))$ such that

$$|\nabla W(t,x)| \leqslant b(t)\varphi(|x|)$$

for all $(t,x) \in R \times R^N$, where $\varphi(s) = O(s^{r_1-1})$ as $s \to 0^+$;
(H_5) There exist an open set $J \subset R$ and two constants $r_3 \in (1,2)$ and $\eta > 0$ such that

$$W(t,x) \geqslant \eta|x|^{r_3}$$

for all $(t,x) \in J \times R^N$, $|x| \leqslant 1$.

Then problem (1) has at least one nontrivial homoclinic orbit. Moreover if.

(H_6) $W(t,-x) = W(t,x)$ for all $(t,x) \in R \times R^N$.

Then problem (1) has infinitely many homoclinic orbits.

The main purpose of this paper is to generalize and improve the results in [19,20,31]. We first prove a new compact embedding theorem under condition (L') and then we obtain the existence and multiplicity of homoclinic orbits of problem (1) by assuming W satisfies a kind of subquadratic condition which is different from the ones in [18–22,25,28,31]. Our main results are the following theorems.

Theorem 1. *Assume that L satisfies (L') and W satisfies the following conditions(W_1) There exist three constants $\delta > 0,\ r_1 \in (1,2),\ s_1 \in \left(1,\frac{2}{2-r_1}\right]$ and a function $a_1 \in L^{s_1}(R,[0,+\infty))$ such that*

$$|\nabla W(t,x)| \leqslant a_1(t)|x|^{r_1-1}$$

for all $t \in R$ and $x \in R^N$ with $|x| \leqslant \delta$;(W_2) There exist three constants $M > 0,\ r_2 \in (1,2),\ s_2 \in \left(1,\frac{2}{2-r_2}\right]$ and a function $a_2 \in L^{s_2}(R,[0,+\infty))$ such that

$$|W(t,x)| \leqslant a_2(t)|x|^{r_2}$$

for all $t \in R$ and $x \in R^N$ with $|x| \geqslant M$;(W_3) For every $m > \delta$, there exist $s_3 > 1$ and $b_m \in L^{s_3}(R,[0,+\infty))$ such that

$$|\nabla W(t,x)| \leqslant b_m(t)$$

for all $t \in R$ and $x \in R^N$ with $|x| \leqslant m$;(W_4) There exist constants $r_4 \in (1,2)$, $\eta > 0$ and $\zeta > 0$ such that

$$W(t,x) \geqslant \eta|x|^{r_4}$$

for all $t \in \Omega$ and $x \in R^N$ with $|x| \leqslant \zeta$, where meas{Ω} > 0.

Then problem (1) possesses a nontrivial homoclinic orbit.

Theorem 2. *Assume that L satisfies (L') and W satisfies (W_1), (W_2), (W_3), (W_4) and (H_6). Then problem (1) has infinitely many homoclinic orbits.*

Remark 1. Theorem 2 unifies and improves Theorems A, B and C. To show this, it suffices to show that Theorem 2 improves Theorem C, for Theorems A and B are special cases of Theorem C. (W_1) and (W_3) can be implied by (H_4) and are real weaker than (H_4). $\nabla W(t,x)$ is globally controlled by $b(t)\varphi(|x|)$ in (H_4), while (W_3) is a local condition. J is an open set in (H_5), while Ω is just assumed to be a set with positive measure in (W_4). In our theorems r_1 and r_2 are separately defined, while r_1 is assumed to be lesser than r_2 in Theorem C. The sets of a_i in our theorems are much larger than the ones in Theorem C. In Theorem C the authors assume $a_i \in L^{\frac{2}{2-r_1}}$, while in our theorems $a_i \in L^{s_i}$ for some $s_i \in (1,\frac{2}{2-r_i}]$. There are some functions L and W which satisfy Theorems 1 and 2, but do not satisfy the corresponding results in [18–22,25,28,31] for example

$$L(t) = I_N, \quad W(t,x) = \tilde{a}(t)|x|^{\frac{3}{2}}, \tag{2}$$