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Waves and statics for functionally graded materials and laminates ☆

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ABSTRACT

It is shown here that, for any laminated stacking of elastic materials which are transversely isotropic with respect to an axis Oz, having elastic moduli which may depend either continuously or discontinuously on the coordinate z, surface-guided disturbances at any frequency are governed by the reduced membrane equation. An analogous treatment of the statics of plates having the same structure, yields the static theory of functionally graded plates due originally to Spencer et al. Thus, for functionally graded transversely isotropic plates with traction-free surfaces, displacements have a structure closely analogous to dynamic disturbances and are represented in terms of solutions to the biharmonic equation.

1. Introduction

The static theory of linearly elastic functionally graded and laminated plates was initiated by Kaprielian et al. [1] and developed by Tony Spencer and co-workers (see [2,3] and references therein). In this theory, the material is, at each point, taken to be isotropic, but with Lamé constants and density depending upon the through-thickness coordinate. More recently, it has been shown [4] that time-harmonic surface waves and plate waves in a transversely isotropic medium may be related to a solution of the reduced membrane equation (the Helmholtz equation in two dimensions). Following the recent [5] recognition that a rotational invariance of material properties is the key to this result, it is now clear that a similar reduction to a scalar equation is possible for waves in all plates and half-spaces which locally are transversely isotropic. The density and elastic moduli may vary continuously or discontinuously as functions of the cartesian coordinate z, so describing functionally graded or laminated media.

Besides recording this generalization of Achenbach's result and of those of Kiselev [6,7] and coworkers, this paper links the dynamic theory to the static theory which owes so much to Tony Spencer. In particular, for traction-free plates formed of any stacking of transversely isotropic, functionally graded materials, displacements are described through a solution to the biharmonic equation and are closely analogous to other solutions found by Spencer [8] and based upon plane and anti-plane strain. The paper also reveals a hierarchy of solutions, as shown by England [9], in which transverse loads are solutions to Laplace's equation, the biharmonic equation, etc.

2. Uni-directional waves in functionally graded materials

Using linear elasticity theory, the components t_{ij} of Cauchy stress are related to the displacement components $u_i(\mathbf{x}, t)$ through $t_{ij} = c_{ijlm}u_{l,m}$, where x_1, x_2 and $x_3 \equiv z$ are Cartesian coordinates, c_{ijlm} are the elastic moduli and $_{,j}$ denotes partial

☆ In memory of Tony Spencer – for more than a quarter of a century an exemplary Head of Department and an inspirational leader in research.

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differentiation with respect to x_j. The moduli c_{ijlm} and density ρ are allowed to be continuous or discontinuous functions of the coordinate z, either in the half-space $z \equiv x_3 > 0$ or within a plate $0 < z < h$. In a uniform half-space $z \geqslant 0$, the standard surface wave (a Rayleigh wave) is a solution of the Euler equation $t_{ij,j} = \rho \ddot{u}_i$ in $z > 0$, the traction-free boundary condition $t_{i3} = 0$ at $z = 0$ and the decay condition $\mathbf{u} \to \mathbf{0}$ as $z \to 0$, with displacements depending on only the depth z and a travelling wave coordinate $x_1 - ct$. Here, a dot denotes partial differentiation with respect to time t and the constant c is the propagation speed. The displacement field and speed c are found by seeking $u_j = \mathscr{R}e\, U_j(z,k)\mathrm{e}^{\mathrm{i}k(x_1-ct)}$, so yielding a system of constant coefficient differential equations. c is then determined (uniquely) so as to ensure compatibility between the traction-free and decay conditions. It has been shown [4,5] how these solutions may be generalized to give solutions in which $u_3 = \mathscr{R}e\, \hat{w}(x_1,x_2)U_3(z,k)\mathrm{e}^{-\mathrm{i}kct}$, where $\hat{w}(x_1,x_2)$ is *any* solution to the reduced membrane equation. Here, following steps used in [5] yields a similar result for transversely isotropic media with Oz as symmetry axis and with density and elastic coefficients being *any piecewise continuous functions* of z.

Let $z = z_p,\ (p = 1,2,\ldots P)$ be the only locations at which either the density and/or the elastic coefficients are discontinuous. The governing system is then

$$t_{ij,j} = \rho \ddot{u}_i \qquad \text{in } z > 0,\ \ z \neq z_p \quad (p = 1,2,\ldots,P), \tag{2.1}$$

with traction-free condition

$$t_{i3} \equiv c_{i3lm}u_{l,m} = 0 \qquad \text{at} \quad z = 0, \tag{2.2}$$

continuity conditions

$$[[t_{i3}]] = 0\,, \quad [[u_i]] = 0 \qquad \text{at } \ z = z_p \tag{2.3}$$

and the decay condition $\mathbf{u} \to \mathbf{0}$ as $z \to \infty$. Here, $[[\ \ ]]$ denotes the jump in a quantity at $z = z_p$. For transversely isotropic materials, the elastic moduli are such that the stress components have the form

$$\begin{aligned} &t_{11} = C_{11}u_{1,1} + C_{12}u_{2,2} + C_{13}u_{3,3}, \qquad && t_{23} = t_{32} = C_{44}(u_{2,3} + u_{3,2}),\\ &t_{22} = C_{12}u_{1,1} + C_{11}u_{2,2} + C_{13}u_{3,3}, && t_{13} = t_{31} = C_{44}(u_{3,1} + u_{1,3}),\\ &t_{33} = C_{13}(u_{1,1} + u_{2,2}) + C_{33}u_{3,3}, && t_{12} = t_{21} = C_{66}(u_{1,2} + u_{2,1}),\\ &\text{with } C_{66} = \frac{1}{2}(C_{11} - C_{12}). \end{aligned} \tag{2.4}$$

Then, seeking travelling waves of the form $\mathbf{u} = \mathscr{R}e\mathbf{U}(z,k)\mathrm{e}^{\mathrm{i}\theta}$, $\mathbf{t} = \mathscr{R}e\mathbf{T}(z,k)\mathrm{e}^{\mathrm{i}\theta}$, where $\theta \equiv kx_1 - \omega t$ yields (with $\mathbf{U} = U(z,k)\mathbf{e}_1 + V(z,k)\mathbf{e}_2 + W(z,k)\mathbf{e}_3$) from (2.1) the ordinary differential equations in $z > 0$ ($z \neq z_p$)

$$[C_{44}(U' + \mathrm{i}kW)]' + \mathrm{i}kC_{13}W' + (\omega^2\rho - k^2C_{11})U = 0, \tag{2.5}$$

$$[C_{44}V']' + (\omega^2\rho - k^2C_{66})V = 0 \tag{2.6}$$

and

$$[C_{33}W' + \mathrm{i}kC_{13}U]' + \mathrm{i}kC_{44}U' + (\omega^2\rho - k^2C_{44})W = 0. \tag{2.7}$$

Here, primes denote ordinary differentiation with respect to z. At each location $z = z_p$, the continuity conditions (2.3) yield

$$\begin{aligned} &[[C_{44}(U' + \mathrm{i}kW)]] = 0 = [[C_{44}V']] = [[C_{33}W' + \mathrm{i}kC_{13}U]],\\ &[[U]] = [[V]] = [[W]] = 0, \end{aligned} \tag{2.8}$$

while, at the traction-free surface $z = 0$, the conditions are

$$C_{44}(U' + \mathrm{i}kW) = 0 = C_{44}V' = C_{33}W' + \mathrm{i}kC_{13}U. \tag{2.9}$$

Also, decay is ensured by choosing $U, V, W \to 0$ as $z \to \infty$. For waves in a plate $0 < z < h$, the decay condition is replaced by Eq. (2.9) at $z = h$.

Either for a half-space $z > 0$ or for a plate, transverse (shear-horizontal) displacements $\mathbf{U} = V(z,k)\mathbf{e}_2$ are seen to uncouple from the in-plane (sagittally-polarized) displacements $\mathbf{U} = U(z,k)\mathbf{e}_1 + W(z,k)\mathbf{e}_3$. In a uniform half-space, the only solutions are the Rayleigh wave, with $V \equiv 0$, with U and W specific linear combinations of two decaying exponentials and with $c \equiv \omega/k$ equal to the Rayleigh wave speed c_R. In a uniform half-space $z > h$ with a dissimilar adjoining layer $0 < z < h$, Love waves have transverse displacement V decaying in $z > h$, but oscillatory in the layer. The explicit solutions

$$V(z) = \begin{cases} \cos \hat{\beta}z & 0 \leqslant z \leqslant h,\\ \cos \hat{\beta}h\ \mathrm{e}^{-\beta(z-h)} & h \leqslant z, \end{cases} \qquad U = W = 0, \tag{2.10}$$

where $\hat{\beta} = \sqrt{(\hat{\rho}\omega^2 - \hat{C}_{66}k^2)/\hat{C}_{44}}$ and $\beta = \sqrt{(C_{66}k^2 - \rho\omega^2)/C_{44}}$ (with hats denoting density and elastic moduli in the layer) show that ω is related to k through the *dispersion relation* $\hat{C}_{44}\hat{\beta}\tan\hat{\beta}h = C_{44}\beta$. Since, $C_{44}\beta^2 = (C_{66} - \rho\hat{C}_{66}/\hat{\rho})k^2 + (\rho\hat{C}_{44}/\hat{\rho})\hat{\beta}^2$, this shows that the waves are dispersive ($\hat{\beta}/k$ depends upon k so that ω/k is not constant). Also, these waves exist in many *modes*, with $V(z,k)$ having $0, 1, 2, \ldots$ zeros within the layer. Moreover, in these same materials, sagitally-polarized waves generalizing Rayleigh waves also exist – with many modes corresponding to a chosen value of k, each being dispersive.