International Journal of Engineering Science 47 (2009) 1331–1338

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Solvability of boundary value problems in a theory of plane-strain elasticity with boundary reinforcement

P. Schiavone *, C.-Q. Ru

Department of Mechanical Engineering, University of Alberta, Edmonton, Alberta, Canada T6G 2G8

ARTICLE INFO

Article history:
Available online 11 October 2008

AMS classification:
74B05
45E05
35J25

Keywords:
Elastic reinforcement
Linear elasticity
Nonstandard boundary condition
Elastic coating
Open arcs

ABSTRACT

We investigate the solvability of a recent mathematical model describing plane deformations of an elastic solid whose boundary is partially reinforced by a thin elastic coating.

1. Introduction

Problems dealing with plane deformations of elastic solids with reinforced or coated boundaries have been dealt with extensively in the literature (see, for example [1–8] and the references contained therein). The main objectives of these analyses is to understand the mechanics of coated surfaces as well as to attempt to simulate the mechanical response of materials subjected to various industrial surface processing techniques such as *shot-peening*. In addition, since these 'reinforced surfaces' essentially incorporate the effects of surface stresses, this class of problems is also of great interest to researchers working in the emerging area of nanomechanics in which the effects of surface stresses have been included in continuum models in an attempt to understand the size-dependency of material properties at the nano-scale (see, for example [9]). These examples provide compelling physical motivation for the formulation of a well-posed mathematical model predicting the mechanical response of materials which incorporate (in some form) the effects of surface reinforcement.

The study of the solvability of a new mathematical model describing a linear theory of plane-strain elasticity with boundary reinforcement was initiated by the authors in [7]. In that paper, the corresponding fundamental boundary value problems are formulated (including a detailed derivation of the reinforcement (boundary) conditions) and solvability results are proved using the boundary integral equation method. This is a crucial step for, without it, there is no guarantee that a solution of the mathematical model actually exists despite the fact that the physics clearly demonstrates such a

* Corresponding author. Tel.: +1 780 492 3638; fax: +1 780 492 2200.
E-mail addresses: P.Schiavone@ualberta.ca (P. Schiavone), C.Ru@ualberta.ca (C.-Q. Ru).

0020-7225/$ - see front matter
doi:10.1016/j.ijengsci.2008.09.003

solution. A priori knowledge of solvability is the basis of numerical solution and is crucial to the correct formulation of any mathematical model. The results in [7], however, are limited to the case when the reinforced part of the boundary consists of a finite number of sufficiently smooth *closed* curves. The more general case in which the reinforced boundary can be represented by the union of a finite number of *open* curves is of considerable practical interest since it allows for the modelling of a much wider class of physical problems involving elastic coatings or the effects of surface stress [3]. For example, the case where a surface is partially reinforced or partially coated with a thin film occurs in many different industrial applications (see, for example [5,6]). Unfortunately, this more general case is associated with an (already) nonstandard boundary condition (characterizing the effect of the reinforcement), this time, posed over *open* arcs as opposed to closed curves. The additional resulting end-point conditions to be satisfied at the ends of each arc preclude the extension of the methods used in [7] as well as the subsequent establishment of critical results on the solvability of the corresponding mathematical model.

In the present work, we draw on results established by the authors in [10], and formulate the corresponding mixed boundary value problems with an alternative (lower order) form of the reinforcement boundary condition. This form is particularly attractive in that it automatically satisfies all end-point conditions and leads itself well to analysis by the boundary integral equation method. In fact, using the formulation from [10], the boundary value problems are shown to reduce to systems of singular integral equations (as opposed to systems of singular integro-differential equations [7]) for which Noether's theorems reduce to Fredholm's theorems. This important fact alone allows us to finally establish the required solvability results for this more general model.

2. Preliminaries

In what follows, Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, we sum over repeated indices, $\mathscr{M}_{m\times n}$ is the space of $(m\times n)$-matrices, E_n is the identity element in $\mathscr{M}_{n\times n}$, a superscript T indicates matrix transposition and $(\cdots)_{,\alpha}\equiv\partial(\cdots)/\partial x_\alpha$. Also, if X is a space of scalar functions and v a matrix, $v\in X$ means that every component of v belongs to X. Let S be a multiply-connected domain in $\mathbb{R}^2$ whose boundary ∂S is described by the union of a finite number of sufficiently smooth closed curves and assumed to be positively oriented in the sense of Green's theorem for the plane. We regard a subset Γ (consisting of the union of a finite number of sufficiently smooth open curves L_i with end-points a_i and b_i $(i=1,\ldots,m)$ such that L_j and L_k have no point in common for $j\neq k$) of ∂S as being coated with a thin elastic film that deforms as a material curve. We assume that S is occupied by a homogeneous and isotropic elastic material with Lamé constants λ and μ. The state of plane-strain is characterized by a displacement field $u=(u_1,u_2,u_3)^{\mathrm{T}}$ of the form

$$u_\alpha=u_\alpha(x_1,x_2),\quad u_3=0, \tag{1}$$

where $x=(x_1,x_2)$ is a generic point in $\mathbb{R}^2$. In the absence of body forces (1) leads to the system of equilibrium equations:

$$L(\partial x)u(x)=0 \tag{2}$$

in which, now, $u=(u_1,u_2)^{\mathrm{T}}, L(\partial x)=L(\partial/\partial x_1,\partial/\partial x_2)$ is the matrix partial differential operator defined by

$$L(\xi_1,\xi_2)=\begin{pmatrix}\mu\Delta+(\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2\\ (\lambda+\mu)\xi_1\xi_2 & \mu\Delta+(\lambda+\mu)\xi_2^2\end{pmatrix}$$

and $\Delta=\xi_1^2+\xi_2^2$. Together with L, we consider the boundary stress operator $T(\partial x)=T(\partial/\partial x_1,\partial/\partial x_2)$ defined by

$$T(\xi_1,\xi_2)=\begin{pmatrix}(\lambda+2\mu)n_1\xi_1+\mu n_2\xi_2 & \mu n_2\xi_1+\lambda n_1\xi_2\\ \lambda n_2\xi_1+\mu n_1\xi_2 & \mu n_1\xi_1+(\lambda+2\mu)n_2\xi_2\end{pmatrix},$$

where $n=(n_1,n_2)^{\mathrm{T}}$ is the unit outward normal to ∂S. With the assumption that

$$3\lambda+2\mu>0,\quad \mu>0, \tag{3}$$

it is clear that the operator L is elliptic and the internal energy density given by

$$E(u,u)=\frac{1}{2}[\lambda(u_{1,1}+u_{2,2})^2+2\mu(u_{1,1}^2+u_{2,2}^2)+\mu(u_{1,2}+u_{2,1})^2]$$

is a positive quadratic form. Further, $E(u,u)=0$ if and only if

$$u(x)=(c_1+c_0x_2,c_2-c_0x_1)^{\mathrm{T}}, \tag{4}$$

where c_0 and c_α are arbitrary constants. Eq. (4) is the most general rigid displacement compatible with the theory of plane-strain. If we write

$$F=\begin{pmatrix}1 & 0 & x_2\\ 0 & 1 & -x_1\end{pmatrix},$$

where the columns $F^{(i)}$ form a basis for (4), then any vector of the form (4) can be written as Fk where $k\in\mathscr{M}_{3\times 1}$ is constant and arbitrary. Further, it is clear that $LF=0$ in $\mathbb{R}^2$ and that $TF=0$ on ∂S.