International Journal of Engineering Science 47 (2009) 1367–1378

Contents lists available at ScienceDirect

International Journal of Engineering Science

journal homepage: www.elsevier.com/locate/ijengsci

Linear theory for the bending and extension of a thin, residually stressed, fiber-reinforced lamina

David J. Steigmann

Department of Mechanical Engineering, University of California Berkeley, CA 94720, USA

ARTICLE INFO

Article history:
Received 3 September 2008
Accepted 10 September 2008
Available online 1 November 2008

To the memory of A.J.M. Spencer

Keywords:
Plate theory
Residual stress
Linear elasticity
Transverse isotropy

ABSTRACT

The Euler equations of a thickness-wise expansion of the potential energy of a thin body, truncated at a specified order in thickness, furnish a model for the bending and stretching of plates and shells. However, truncated expansions of the energy typically do not lead to well-posed minimization problems. This is related to the fact that the truncations may fail to satisfy the relevant Legendre–Hadamard condition, which is necessary for the existence of minimizers. This lack of well-posedness is thus entirely consistent with well-posedness in the exact theory. However, it is an inconvenience from the viewpoint of analysis. What is desired is an accurate, well-posed truncation that preserves the structure of classical plate theory. The present work is concerned with the development of such a model for a uniform fiber-reinforced lamina.

1. Introduction

Approximate theories for the bending and stretching of fiber-reinforced laminates are a cornerstone of structural analysis and are well described in many texts and monographs, while Rogers et al [1], building on Michell's [2] ideas, have made fundamental advances in the exact theory. However, to date virtually no work has been done on the rigorous derivation of approximate models amenable to engineering calculations. An important exception is the recent work of Paroni [3], in which a rigorous leading order (in thickness) model is obtained for a single lamina possessing reflection symmetry with respect to its midplane. The proof relies on sophisticated concepts in functional analysis which may not be well known to those more concerned with applications of plate theory. Indeed, the recent literature on modern developments in plate theory (e.g. [4]) may give the impression of a recondite subject. In contrast, we show here that straightforward reasoning yields not only the correct model, but also a simpler framework in which it can be interpreted and extended. In particular, we recover Paroni's model, specialized to laminae with fiber symmetry, without resorting to functional analysis.

Standard notation is used throughout. Thus, we use bold face for vectors and tensors and indices to denote their components. Latin indices take values in $\{1, 2, 3\}$; Greek in $\{1, 2\}$. The latter are associated with surface coordinates and associated vector and tensor components. A dot between bold symbols is used to denote the standard inner product. Thus, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are second-order tensors, then $\mathbf{A}_1 \cdot \mathbf{A}_2 = tr(\mathbf{A}_1\mathbf{A}_2^t)$, where $tr(\cdot)$ is the trace and the superscript ^{t} is used to denote the transpose. The norm of a tensor $\mathbf{A}$ is $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$. The linear operator $Sym(\cdot)$ delivers the symmetric part of its second-order tensor argument. The notation $\otimes$ identifies the standard tensor product of vectors. If $\mathbf{C}$ is a fourth-order tensor, then $\mathbf{C}[\mathbf{A}]$ is the second-order tensor with orthogonal components $C_{ijkl}A_{kl}$. Finally, we use symbols such as Div and D to denote the three-dimensional divergence and gradient operators, while div and ∇ are reserved for their two-dimensional counterparts. Thus, for example, $Div\mathbf{A} = A_{ij,j}\mathbf{e}_i$ and $div\mathbf{A} = A_{i\alpha,\alpha}\mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is an orthonormal basis and subscripts preceded by commas are used to denote partial derivatives with respect to Cartesian coordinates.

E-mail address: steigman@newton.berkeley.edu

0020-7225/$ - see front matter
doi:10.1016/j.ijengsci.2008.09.004

The three-dimensional equation of equilibrium without body force is

$$Div\widetilde{\mathbf{P}} = \mathbf{0}, \tag{1}$$

where

$$\widetilde{\mathbf{P}} = \mathbf{S} + \widetilde{\mathbf{H}}\mathbf{S} + \mathbf{C}[\widetilde{\mathbf{H}}]. \tag{2}$$

is the linear approximation to the Piola stress, $\mathbf{S}$ is the (symmetric) residual stress, $\widetilde{\mathbf{H}} = D\tilde{\mathbf{u}}$ is the displacement gradient, $\tilde{\mathbf{u}}(\mathbf{x})$ is the three-dimensional displacement field and $\mathbf{C}$ is the fourth-order tensor of elastic moduli. We require the latter to possess the minor symmetries

$$\mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2] = \mathbf{A}_1^t \cdot \mathbf{C}[\mathbf{A}_2],\ \ \mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2] = \mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2^t] \tag{3}$$

and the major symmetry

$$\mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2] = \mathbf{A}_2 \cdot \mathbf{C}[\mathbf{A}_1] \tag{4}$$

for all second-order tensors $\mathbf{A}_1, \mathbf{A}_2$. These restrictions in turn ensure that

$$\widetilde{\mathbf{P}} = U_{\widetilde{\mathbf{H}}}, \tag{5}$$

where

$$U(\widetilde{\mathbf{H}}; x) = \mathbf{S} \cdot \widetilde{\mathbf{H}} + \frac{1}{2}(\widetilde{\mathbf{H}}\mathbf{S} \cdot \widetilde{\mathbf{H}} + \widetilde{\mathbf{H}} \cdot \mathbf{C}[\widetilde{\mathbf{H}}]) \tag{6}$$

is the quadratic-order approximation to the strain energy per unit volume of R in which explicit dependence on $\mathbf{x} \in R$ is present if the material is non-uniform. Such dependence occurs through the residual stress and the moduli. In this work we take these to be independent of $\mathbf{x}$ and thus restrict attention to uniform materials.

This expression is consistent with that obtained by Hoger [5], who notes that the linear term may be discarded if the residual stress is required to be self-equilibrating in the sense that $Div\mathbf{S}$ vanishes in the interior of the body and $\mathbf{Sn}$ vanishes on a part of its boundary (with exterior unit normal $\mathbf{n}$) where null residual stress is prescribed, the displacement being assigned on the complementary part. That this is so follows easily from $\mathbf{S} \cdot \widetilde{\mathbf{H}} = Div(\mathbf{S}\tilde{\mathbf{u}}) - \tilde{\mathbf{u}} \cdot Div\mathbf{S} = Div(\mathbf{S}\tilde{\mathbf{u}})$. The volume integral of this term is expressible as the surface integral of $\tilde{\mathbf{u}} \cdot \mathbf{Sn}$ over the boundary; this vanishes where null residual tractions are specified and performs null working on the remainder. Accordingly, the first term in (6) contributes only a disposable constant to the energy of the body.

We impose the strong-ellipticity condition

$$(\mathbf{w} \cdot \mathbf{Sw})\mathbf{v} \cdot \mathbf{v} + \mathbf{v} \otimes \mathbf{w} \cdot \mathbf{C}[\mathbf{v} \otimes \mathbf{w}] > 0 \quad \text{for all} \quad \mathbf{v} \otimes \mathbf{w} \neq 0. \tag{7}$$

This is necessary for the undeformed body to be a minimizer of the total strain energy. It is easy to show that the symmetric part of $\mathbf{v} \otimes \mathbf{w}$ vanishes if and only if $\mathbf{v} \otimes \mathbf{w}$ vanishes, so that (7) is meaningful. We also assume this inequality to hold whether or not residual stress is present. Thus we impose the condition

$$\mathbf{v} \otimes \mathbf{w} \cdot \mathbf{C}[\mathbf{v} \otimes \mathbf{w}] > 0 \quad \text{for all} \quad \mathbf{v} \otimes \mathbf{w} \neq 0, \tag{8}$$

which is stronger (resp. weaker) than (7) if the residual stress is positive (resp. negative) definite.

Following Spencer [6] we model the lamina as a transversely isotropic solid. The axes of transverse isotropy are coincident with the direction fields of the (straight) fiber trajectories. The components of $\mathbf{C}$ relative to an orthonormal basis $\{\mathbf{e}_i\}$ are (see Spencer [7])

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_T(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \alpha(\delta_{ij}m_k m_l + m_i m_j \delta_{kl}) + (\mu_L - \mu_T)(m_i m_k \delta_{jl} + m_i m_l \delta_{jk} + m_j m_k \delta_{il} + m_j m_l \delta_{ik}) + \beta m_i m_j m_k m_l, \tag{9}$$

where δ_{ij} is the Kronecker delta; $\alpha, \beta, \lambda, \mu_T$ and μ_L are material constants; and the unit vector $\mathbf{m}$, with components m_i, is the fiber axis, assumed here to be uniform. Spencer [7] shows that μ_T is the shear modulus for shearing in planes transverse to $\mathbf{m}$, whereas μ_L is the shear modulus for shearing parallel to $\mathbf{m}$. The remaining material constants in (9) may be interpreted in terms of extensional moduli and Poisson ratios [7]. In this model the fibers are modelled as perfectly flexible curves that transmit conventional stresses. In particular, couple stresses, generated by the flexural stiffness of the fibers, are ignored. This is in accord with standard laminate theory. However, an extension of the three-dimensional theory that accounts for fiber bending stiffness is available [8], and would no doubt lead to useful advances in the development of plate theory.

The general form of the residual stress may be derived by enumerating the strain invariants for transverse isotropy that are linear in the (infinitesimal) strain. These are [7] $\mathbf{I} \cdot \mathbf{H}$ and $\mathbf{m} \otimes \mathbf{m} \cdot \mathbf{H}$. Comparison with the linear term in (6) then furnishes

$$\mathbf{S} = S_T(\mathbf{I} - \mathbf{m} \otimes \mathbf{m}) + S_L\mathbf{m} \otimes \mathbf{m}, \tag{10}$$

where S_T is the constant residual stress in the isotropic plane and S_L is the constant residual uniaxial stress along $\mathbf{m}$.