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Bach flow

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ABSTRACT

In this paper, we study the Bach flow which is defined as

$$\frac{\partial}{\partial t}g_{ij} = -B_{ij}$$

where B_{ij} is the Bach tensor. Among other things, we study the solitons to the Bach flow. We also study the Bach flow on a four-dimensional Lie group, in which we study the convergence of the Bach flow.

1. Introduction

In an n-dimensional Riemannian manifold (M^n, g), $n \geq 4$, the Bach tensor, introduced by Bach in [1], is defined as

$$B_{ij} = \frac{1}{n-3}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}R_{kl}W_{ikjl}. \tag{1.1}$$

Hereafter, we use the Einstein summation convention: we sum over the repeat indices. Hence, (1.1) should be read as

$$B_{ij} = \frac{1}{n-3}\sum_{k,j=1}^{n}\nabla_k\nabla_l W_{ikjl} + \frac{1}{n-2}\sum_{k,j=1}^{n}R_{kl}W_{ikjl}.$$

Here

$$\begin{aligned} W_{ikjl} = R_{ikjl} &- \frac{1}{n-2}(R_{ij}g_{kl} + R_{kl}g_{ij} - R_{il}g_{kj} - R_{kj}g_{il}) \\ &+ \frac{R}{(n-1)(n-2)}(g_{ij}g_{kl} - g_{il}g_{kj}) \end{aligned} \tag{1.2}$$

is the Weyl tensor, R_{ikjl} is the Riemann curvature tensor, R_{ij} is the Ricci curvature tensor, and R is the scalar curvature of the metric g. It is easy to see that if (M^n, g) is either locally conformally flat (i.e. $W_{ikjl} = 0$) or Einstein, then (M^n, g) is *Bach-flat*: $B_{ij} = 0$. See Proposition 2.2 when g is Einstein.

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The case when $n = 4$ is the most interesting, as it is well known that (see [2]) on any compact 4-manifold (M^4, g), Bach-flat metrics are precisely the critical point of the conformally invariant functional on the space of metrics,

$$\mathcal{W}(g) = \int_M |W|_g^2 dV_g$$

where W denotes the Weyl tensor of g. Moreover, if (M^4, g) is either half conformally flat (i.e. self-dual or anti-self-dual) or locally conformal to an Einstein manifold, then its Bach tensor vanishes.

A geometric flow which was defined using the Bach tensor has been introduced in [3]: the Bach flow is defined as follows:

$$\frac{\partial}{\partial t} g_{ij} = -B_{ij}. \tag{1.3}$$

In [3], Das and Kar studied the Bach flow on some product manifolds equipped with the product metric.

In this paper, we study the Bach flow more generally. After collecting some properties of the Bach flow in Section 2, we study in Section 3 the solitons to the Bach flow. In particular, we prove that any compact soliton to the Bach flow must be Bach-flat. See Theorems 3.1 and 3.2. Finally, in Section 4, we study the Bach flow on a four-dimensional Lie groups, in which we consider the convergence of the Bach flow.

2. Some properties of Bach flow

In this section, we collect some properties of Bach flow. Along the Bach flow (1.3), we have

$$\frac{\partial}{\partial t} dV_g = \frac{1}{2} g^{ij} B_{ij} dV_g = 0, \tag{2.1}$$

since the Bach tensor is trace-free. Therefore, we have the following:

Proposition 2.1. *The Bach flow (1.3) preserves the volume of M.*

Note that the Bach flow does not preserve the conformal structure in general. Indeed, the Bach flow preserves the conformal structure only if the initial metric is Bach flat. To see this, note that if $\tilde{g} = e^{2u} g$, then

$$\widetilde{B}_{ij} = e^{-2u} B_{ij} \tag{2.2}$$

where $\widetilde{B}_{ij}$ and B_{ij} are the Bach tensors of $\tilde{g}$ and g respectively. Therefore, if $\tilde{g} = e^{2u} g$ is the solution of the Bach flow, we have

$$2e^{2u} \frac{\partial u}{\partial t} g_{ij} = e^{-2u} B_{ij} \tag{2.3}$$

by (1.3) and (2.2). Taking trace of both sides in (2.3), we get

$$\frac{\partial u}{\partial t} = 0 \tag{2.4}$$

since the Bach tensor is trace-free by the definition in (1.1). It follows from (2.4) that $\frac{\partial}{\partial t} \tilde{g}_{ij} = 0$, which implies that $\widetilde{B}_{ij} = 0$ for all $t \geq 0$. In particular, the initial metric is Bach flat.

The following proposition is well-known.

Proposition 2.2. *If g is Einstein, then g must be Bach flat.*

To be self-contained, we give the proof here.

Proof of Proposition 2.2. If g is Einstein, then

$$R_{ij} = \frac{R}{n} g_{ij}. \tag{2.5}$$

In particular, the scalar curvature R must be constant since $n \geq 4$. It follows from (2.5) that Weyl tensor can be rewritten as

$$\begin{aligned} W_{ikjl} &= R_{ikjl} - \frac{1}{n-2}(R_{ij}g_{kl} + R_{kl}g_{ij} - R_{il}g_{kj} - R_{kj}g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ij}g_{kl} - g_{il}g_{kj}) \\ &= R_{ikjl} - \frac{R}{n(n-1)}(g_{ij}g_{kl} - g_{il}g_{kj}). \end{aligned} \tag{2.6}$$

It follows from (1.2) and (2.5) that

$$R_{kl} W_{ikjl} = \frac{R}{n} g_{kl} W_{ikjl} = 0. \tag{2.7}$$