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Abundant lumps and their interaction solutions of (3+1)-dimensional linear PDEs

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ABSTRACT

The paper aims to explore the existence of diverse lump and interaction solutions to linear partial differential equations in (3+1)-dimensions. The remarkable richness of exact solutions to a class of linear partial differential equations in (3+1)-dimensions will be exhibited through Maple symbolic computations, which yields exact lump, lump-periodic and lump–soliton solutions. The results expand the understanding of lump, freak wave and breather solutions and their interaction solutions in soliton theory.

1. Introduction

Lump solutions are a particular kind of exact solutions, which describe various important nonlinear phenomena in nature [1,2]. More specifically, such solutions can be generated from solitons by taking long wave limits [3]. There are also positons and complexitons to integrable equations, enriching the diversity of solitons [4,5]. Interaction solutions between two different kinds of exact solutions exhibit more diverse nonlinear phenomena [6].

Soliton solutions are exponentially localized in all directions in space and time, and lump solutions, rationally localized in all directions in space. Through a Hirota bilinear form:

$$P(D_x, D_t)f \cdot f = 0, \tag{1.1}$$

where P is a polynomial and D_x and D_t are Hirota's bilinear derivatives, an N-soliton solution in (1+1)-dimensions can be defined by

$$f = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i \xi_i + \sum_{i<j} \mu_i \mu_j a_{ij}\Big), \tag{1.2}$$

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where

$$\begin{cases} \xi_i = k_i x - \omega_i t + \xi_{i,0}, \ 1 \le i \le N, \\ e^{a_{ij}} = -\dfrac{P(k_i - k_j, \omega_j - \omega_i)}{P(k_i + k_j, \omega_j + \omega_i)}, \ 1 \le i < j \le N, \end{cases} \tag{1.3}$$

with k_i and ω_i satisfying the dispersion relation and $\xi_{i,0}$ being arbitrary shifts. The KPI equation

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0 \tag{1.4}$$

has a lump solution [7]:

$$u = 2(\ln f)_{xx}, \ f = \big(a_1 x + a_2 y + a_3 t + a_4\big)^2 + \big(a_5 x + a_6 y + a_7 t + a_8\big)^2 + a_9, \tag{1.5}$$

where

$$a_3 = \frac{a_1 a_2{}^2 - a_1 a_6{}^2 + 2\,a_2 a_5 a_6}{a_1{}^2 + a_5{}^2}, \ a_7 = \frac{2a_1 a_2 a_6 - a_2{}^2 a_5 + a_5 a_6{}^2}{a_1{}^2 + a_5{}^2}, \ a_9 = \frac{3(a_1{}^2 + a_5{}^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \tag{1.6}$$

and the other parameters a_i's are arbitrary but need to satisfy $a_1 a_6 - a_2 a_5 \neq 0$, which guarantees rational localization in all directions in the (x, y)-plane. Other integrable equations, possessing lump solutions, include the three-dimensional three-wave resonant interaction [8], the BKP equation [9,10], the Davey–Stewartson equation II [3], the Ishimori-I equation [11] and many others [12,13].

It is recognized by making symbolic computations that many nonintegrable equations possess lump solutions as well, including (2+1)-dimensional generalized KP, BKP and Sawada–Kotera equations [14–16]. Moreover, various studies show the existence of interaction solutions between lumps and another kind of exact solutions to nonlinear integrable equation in (2+1)-dimensions, which contain lump–soliton interaction solutions (see, e.g., [17–20]) and lump–kink interaction solutions (see, e.g., [21–24]). Nevertheless, in the (3+1)-dimensional case, only lump-type solutions are presented for the integrable Jimbo–Miwa equations, which are rationally localized in almost all but not all directions in space. All presented analytical rational solutions to the (3+1)-dimensional Jimbo–Miwa equation in [25–27] and the (3+1)-dimensional Jimbo–Miwa like equation in [28] are not rationally localized in all directions in space. It is absolutely very interesting and important to explore lump and interaction solutions to partial differential equations in (3+1)-dimensions.

This paper aims at showing that there do exist abundant lump solutions and their interaction solutions to linear partial differential equations in (3+1)-dimensions. A class of particular examples in (3+1)-dimensions will be considered to exhibit such solution phenomena. We will explicitly generate lump solutions and mixed lump-periodic and lump–soliton solutions for a specially chosen class of (3+1)-dimensional linear partial differential equations. Based on Maple symbolic computations, sufficient conditions and examples of lump and interaction solutions will be provided, together with three-dimensional plots and contour plots of special examples of the presented solutions. Some concluding remarks will be given in the final section.

2. Abundant lump and interaction solutions

Let $u = u(x, y, z, t)$ be a real function of $x, y, z, t \in \mathbb{R}$. We consider a class of linear partial differential equations (PDEs) in (3+1)-dimensions:

$$\alpha_1 u_{xy} + \alpha_2 u_{xz} + \alpha_3 u_{xt} + \alpha_4 u_{yz} + \alpha_5 u_{yt} + \alpha_6 u_{zt} + \alpha_7 u_{xx} + \alpha_8 u_{yy} + \alpha_9 u_{zz} + \alpha_{10} u_{tt} = 0, \tag{2.1}$$

where $\alpha_i, \ 1 \le i \le 10$, are real constants, and the subscripts denote partial differentiation.

We search for a kind of exact solutions

$$u = v(\xi_1, \xi_2, \xi_3, \xi_4) \tag{2.2}$$

where v is an arbitrary real function, and $\xi_i, \ 1 \le i \le 4$, are four wave variables:

$$\xi_i = a_i x + b_i y + c_i z + d_i t + e_i, \ 1 \le i \le 4, \tag{2.3}$$

in which a_i, b_i, c_i, d_i and e_i, $1 \le i \le 4$, are real constants to be determined. Then, the linear PDE (2.1) becomes

$$\sum_{i=1}^{4} \sum_{j=i}^{4} w_{ij} v_{\xi_i \xi_j} = 0, \tag{2.4}$$

where w_{ij}, $1 \le i \le j \le 4$, are quadratic functions of the parameters a_i, b_i, c_i and d_i, $1 \le i \le 4$. Upon setting all coefficients of the ten second partial derivatives of v to be zero, we obtain a system of equations on the parameters:

$$\begin{cases} \alpha_1 a_i b_i + \alpha_2 a_i c_i + \alpha_3 a_i d_i + \alpha_4 b_i c_i + \alpha_5 b_i d_i \\ \quad + \alpha_6 c_i d_i + \alpha_7 a_i^2 + \alpha_8 b_i^2 + \alpha_9 c_i^2 + \alpha_{10} d_i^2 = 0, \ 1 \le i \le 4, \\ \alpha_1(a_i b_j + a_j b_i) + \alpha_2(a_i c_j + a_j c_i) + \alpha_3(a_i d_j + a_j d_i) + \alpha_4(b_i c_j + b_j c_i) + \alpha_5(b_i d_j + b_j d_i) \\ \quad + \alpha_6(c_i d_j + c_j d_i) + 2\alpha_7 a_i a_j + 2\alpha_8 b_i b_j + 2\alpha_9 c_i c_j + 2\alpha_{10} d_i d_j = 0, \ 1 \le i < j \le 4. \end{cases} \tag{2.5}$$