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Analytical solutions for a surface-loaded isotropic elastic layer with surface energy effects

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A R T I C L E I N F O

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A B S T R A C T

Consideration of surface (interface) energy effects on the elastic field of a solid material has applications in several modern problems in solid mechanics. The Gurtin–Murdoch continuum model [M.E. Gurtin, A.I. Murdoch, Arch. Ration. Mech. Anal. 57 (1975) 291–323; M.E. Gurtin, J. Weissmuller, F. Larché, Philos. Mag. A 78 (1998) 1093–1109] accounting for surface energy effects is applied to analyze the elastic field of an isotropic elastic layer bonded to a rigid base. The surface properties are characterized by the residual surface tension and surface Lame constants. The general solutions of the bulk medium expressed in terms of Fourier integral transforms and Hankel integral transforms are used to formulate the two-dimensional and axisymmetric three-dimensional problems, respectively. The generalized Young–Laplace equation for a surface yields a set of non-classical boundary conditions for the current class of problems. An explicit analytical solution is presented for the elastic field of a layer. The layer solution is specialized to obtain closed-form solutions for semi-infinite domains. Selected numerical results are presented to show the influence of surface elastic constants and layer thickness on stresses and displacements.

1. Introduction

There is growing interest in the study of the mechanics of nano-scale structures and devices. The natural approach is to consider atomistic modeling techniques for nano-scale domains but such techniques require a very large computational effort. The application of continuum-based approaches is considered attractive due to their lesser complexity and computational efficiency. The surface-to-volume ratio of a nano-scale domain is relatively high compared to that of macro-scale domains. The energy associated with atoms at or near a free surface is different from that of atoms in the bulk. The effect of surface free energy therefore becomes important in the case of nano-scale problems [1]. Povstenko [2] observed that the stress field caused by heterogeneous surface tension in a solid half-space can be used to explain the high stresses in the surface layer that cause a zone with high dislocation density when a surface-active melt interacts with a metal. In addition, for some soft solids, such as polymer gels, the surface energy (hence surface stresses) has an important influence on surface topographical patterns that are used for applications in surface self-assembly regulation, micro-fluidic flow control and direction, etc. [3,4]. Consequently, the study of the elastic field of a solid with surface energy effects is of interest to many current technological developments.

Surface energy effects are generally ignored in traditional continuum mechanics. This is not the case for nano-scale structures due to their high surface/volume ratio, soft materials where the ratio of surface energy per unit area to the bulk Young's

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modulus is comparable to the characteristic size of the material element and other situations where surface tension gradients and other surface energy driven effects have a significant influence on the response. Gurtin and Murdoch [5,6] developed a theoretical framework based on continuum mechanics concepts that included the effects of surface and interfacial energy, in which the surface is modeled as a mathematical layer of zero thickness perfectly bonded to an underlying bulk. The surface (interface) has its own properties and processes that are different from the bulk. Miller and Shenoy [7] and Shenoy [8] demonstrated that size-dependent behaviour of nano-scale structural elements can be modeled by applying the Gurtin–Murdoch continuum model with surface properties determined from atomistic modeling [9]. Tian and Rajapakse [10] examined the size-dependent elastic field due to a nano-scale elliptical defect in an isotropic matrix and observed unstable defect geometries.

The elastic field of a surface-loaded layer of nano-scale thickness bonded to a rigid base has important applications in the study of nano-electronics devices, coatings and films, deformations due to quantum dots, etc. Similarly, the response of a soft elastic layer with surface energy effects can be used in the study of adsorption of molecules/cells into thin layers and their interaction energies, micro-fluidic devices, etc. The classical elasticity solution of a layer of finite thickness bonded to a rigid base was given by Pickett [11] which has found extensive applications in tribology, geomechanics, biomechanics, etc. Povstenko [2] derived the elastic field of a half-space cased by a jump in the surface tension over a circular area by neglecting the bulk properties He and Lim [12] derived the surface Green's functions of a soft *incompressible* isotropic elastic half-space with surface energy effects by using the Gurtin–Murdoch model. In addition to the incompressibility, they further restricted their derivation to the special case where the surface elastic properties are same as the bulk properties. Wang and Feng [13] studied the response of a half-plane subjected to surface pressures by neglecting the surface elastic constants and considering only the influence of constant surface tension. Huang and Yu [14] considered a surface-loaded half-plane with non-zero surface elastic constants in the absence of any surface tension.

In this paper, the fundamental problem of a compressible isotropic elastic layer with complete surface stress effects (non-zero surface tension and surface elastic properties) that is bonded to a rigid base and subjected to surface loading is considered. Both two-dimensional plane and axisymmetric problems are considered. The Gurtin–Murdoch model is applied to derive the elastic field of the layer. Fourier and Hankel integral transforms are used to solve the boundary-value problems involving non-classical boundary conditions associated with the generalized Young–Laplace equation. Closed-form analytical solutions are presented for the case of a layer of infinite thickness (half-plane/space) and in this case the influence of surface energy effects can be explicitly identified. For a layer of finite thickness, the elastic field is examined numerically to assess the influence of surface energy effects and layer thickness.

2. Governing equations and general solutions

Consider an elastic layer of finite thickness bonded to a rigid base as shown in Fig. 1. The layer is subjected to surface loading and its response is modeled by using the Gurtin–Murdoch continuum model [5,6]. According to this model, the surface energy effects are accounted for by considering the surface as a mathematical layer of zero thickness with relevant elastic properties and residual surface tension, that is perfectly bonded to the underlying bulk material. In the bulk, the governing equations are same as those in the classical elasticity. In addition, on a surface (or interface), the generalized Young–Laplace equation [2] and a set of constitutive relations have to be satisfied. The basic equations for small displacements and infinitesimal strains of a continuum with surface stress effects are summarized below based on Gurtin et al. [6].

In the absence of body forces, the three-dimensional equilibrium and constitutive equations of the bulk material are,

$$\sigma_{ij,j} = 0 \tag{1}$$
$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{2}$$

and the classical strain–displacement relationship is,

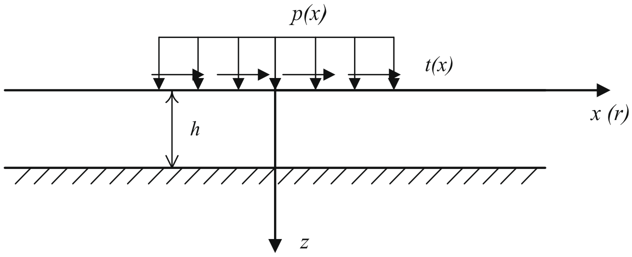

Fig. 1. Elastic layer subjected to surface loading.